Latent Class Analysis: The Empirical Study of Latent Types, Latent Variables, and Latent Structures, and Some Notes on the History of This Subject[1]

Leo A. Goodman
University of California at Berkeley

"Things are seldom what they seem.
Skim milk masquerades as cream."

W. S. Gilbert of Gilbert and Sullivan, *H.M.S. Pinafore*

"Judge not according to appearance."

The Gospel According to St. John

1. INTRODUCTION

I begin this introductory section on latent class analysis by considering this subject in its simplest context; viz., the cross-classification of two dichotomous variables, say, variables A and B. In this context, we have the simple two-way 2×2 cross-classification table $\{A, B\}$, where the two rows of the 2×2 table correspond to the two classes of the dichotomous variable A, and the two columns of the 2×2 table correspond to the two classes of the dichotomous variable B. We let P_{ij} denote the probability that an observation will fall in the i-th row $(i = 1, 2)$ and j-th column $(j = 1, 2)$ of this 2×2 table. In other words, P_{ij} is the probability that an observation will be in the i-th class $(i = 1, 2)$ on variable A and in the j-th class $(j = 1, 2)$ on variable B. When variables A and B are statistically independent of each other, we have the simple relationship

$$P_{ij} = P_i^A P_j^B, \tag{1}$$

where P_i^A is the probability that an observation will fall in the i-th row of the 2×2 table, and P_j^B is the probability that an observation will fall in the j-th column of the 2×2 table. In other words, P_i^A is the probability that an observation will be in the i-th class on variable A, and P_j^B is the probability that an observation will be in the j-th class on variable B; with

$$P_i^A = P_{i+} = \sum_j P_{ij}, \qquad P_j^B = P_{+j} = \sum_i P_{ij}. \tag{2}$$

When variables A and B are not statistically independent of each other (i.e., when formula (1) does not hold true), which is often the case in many areas of empirical research (when both variables A and B are of substantive interest), the researcher analyzing the data in the 2×2 table will usually be interested in measuring the non-independence between the two variables (A and B); and there are many different measures of this non-independence. (Even for the simple 2×2 table, there are many such measures.) However, all of these measures of non-independence (or almost all of them) are deficient in an important respect. Although these measures of non-independence may help the researcher to determine the magnitude of the non-independence between the two variables (A and B), they can *not* help him/her to determine whether or not this non-independence is spurious. In other words, none (or almost none) of the usual measures of the non-independence between variables A and B can help the researcher to determine whether or not the observed relationship (the non-independence) between variables A and B can be explained away by some other variable, say, variable X, where this variable X may be unobserved or unobservable, or latent. Is there a latent variable X that can explain away the observed (manifest) relationship between variables A and B, when we take into account the (unobserved) relationship that this latent variable X may have with variable A and the (unobserved) relationship that the latent variable may have with variable B? The use of latent class models can help the researcher to consider such questions.

The latent variable X introduced above can be viewed as a possible explanatory variable. It can be used at times to explain away the observed relationship between variables A and B even when this observed relationship between the two observed variables (A and B) is statistically significant. At other times, the explanatory latent variable X can be used to help the researcher to explain more fully (rather than to explain away) the observed relationship between the two observed variables. With some sets of data, an appropriate latent class model might include several latent variables as explanatory variables, useful in helping the researcher to explain more fully (or to explain away) the observed relationships among the set of observed variables under consideration. Use of such latent class models can help the researcher in many ways, as we shall see later in this exposition on the use of latent class models and in the chapters that follow in this book on latent class analysis.

The problem of measuring the relationship (the non-independence) between two (or more) observed dichotomous (or polytomous) variables has a long history. This problem has been considered by many researchers in many fields of inquiry at various times throughout the twentieth century; and it is a topic that was also considered by some eminent scholars in the nineteenth century. The use of latent class models as a tool to help researchers gain a deeper understanding of the observed relationships among the observed dichotomous (or polytomous) variables has, on the other hand, a much shorter history in the twentieth century; but it might be worthwhile to note here that some mathematical models that were used earlier in some nineteenth century work can now be viewed as special cases of latent class models or of other kinds of latent structures. With respect to these nineteenth-century models, we refer, in particular, to some work by C. S. Peirce, the great philosopher and logician, and also able scientist and mathematician. (In addition to the recognition he has received for some of his other work, he is also sometimes referred to as the "founder of pragmatism".) Peirce introduced such a model (i.e., a latent structure) in order to gain further insight into the relationship between two observed dichotomous variables in the context of measuring the success of predictions (Peirce 1884; Goodman and Kruskal 1959). We shall return to this example in a later section herein.

The main development of latent class models has taken place during the last half of this past century; and the practical application of these models by researchers in various fields of inquiry has become a realistic possibility only during the last quarter of this past century (after more efficient and more usable statistical methods were developed and more general latent class models were introduced). While the problem of measuring the relationship (the non-independence) between two or more observed dichotomous (or polytomous) variables has arisen and has been considered in many fields of inquiry at various times throughout this past century and in the preceding century, we can expect that researchers in some of these fields of inquiry (and in other fields as well) will find that the introduction and application of latent class models can help them to gain further insight into the observed relationships among these observed variables of interest. The introduction of latent class models can insert a useful perspective into the study of the relationships among these variables.

We have focused our attention so far in this introductory section on latent class analysis on the possible use of a latent dichotomous or polytomous variable (or a set of such latent dichotomous or polytomous variables) as an explanatory variable (or as explanatory variables) in the study of the relationships among a set of observed (or manifest) dichotomous or polytomous variables. (In this case, our primary focus is on the set of observed variables and on possible explanations of the observed relationships among these variables.) We can also use the latent class models in the situation where the observed dichotomous or polytomous variables may be viewed as indicators or markers for an unobserved latent variable X, where the unobserved variable is, in some sense, being measured (in an indirect way and with measurement error) by the observed variables. (In this case, our primary focus is on the unobserved latent variable; and the observed variables are, in some sense, ascriptive or attributive variables pertaining to the latent variable.) And we can also use models of this kind in the study of the relationships among a set of unobserved (or latent) dichotomous or polytomous variables in the situation in which there are observed dichotomous or polytomous variables that can be viewed as indicators or markers for the unobserved (or latent) variables. (In this case, our primary focus is on the set of unobserved latent variables and on the unobserved relationships among these variables.)

2. THE LATENT CLASS MODEL

We shall consider now the latent class model in the situation where variable A is an observed (or manifest) dichotomous or polytomous variable having I classes $(i = 1, 2, \ldots, I)$, variable B is an observed (or manifest) dichotomous or polytomous variable having J classes $(j = 1, 2, \ldots, J)$, and variable X is an unobserved (or latent) dichotomous or polytomous variable having T classes $(t = 1, 2, \ldots, T)$. Let π_{ijt}^{ABX} denote the joint probability that an observation is in class i on variable A, in class j on variable B, and in class t on variable X; and let $\pi_{it}^{\bar{A}X}$ denote the conditional probability that an observation is in class i on variable A, given that the observation is in class t on variable X; and let $\pi_{jt}^{\bar{B}X}$ denote the conditional probability that an observation is in class j on variable B, given that the observation is in class t on variable X; and let π_t^X denote the probability that an observation is in class t on variable X. The latent class model in this situation can be expressed simply as follows:

$$\pi_{ijt}^{ABX} = \pi_t^X \pi_{it}^{\bar{A}X} \pi_{jt}^{\bar{B}X}, \text{ for } i = 1, \ldots, I; \; j = 1, \ldots, J; \; t = 1, \ldots, T. \tag{3}$$

This model states that variables A and B are conditionally independent of each other, given the class level on variable X; i.e.,

$$\pi_{ijt}^{\bar{A}\bar{B}X} = \pi_{ijt}^{ABX} / \pi_t^X = \pi_{it}^{\bar{A}X} \pi_{jt}^{\bar{B}X}, \tag{4}$$

where $\pi_{ijt}^{\bar{A}\bar{B}X} = \pi_{ijt}^{ABX} / \pi_t^X$ is the conditional probability that an observation is in class i on variable A and in class j on variable B, given that the observation is in class t on variable X.

We have presented the latent class model above for the situation in which there are only two observed (manifest) variables (say, A and B). This we do for expository purposes in this introductory section on the latent class model in order to consider this subject in its simplest context. On the other hand, it should be noted that some special problems arise when latent class models are considered in the situation in which there are only two observed variables that do not arise in the situation in which there are more than two observed variables. But these problems need not deter us here. For illustrative purposes, we shall consider next some examples in which latent class models are applied in the analysis of cross-classified data in the situation in which there are two observed variables and also in the situation in which there are more than two observed variables.

3. OUR FIRST EXAMPLE. THE ANALYSIS OF THE RELATIONSHIP BETWEEN TWO OBSERVED VARIABLES

I shall begin this section by considering the cross-classified data presented in Table 1. These data on the relationship between parental socioeconomic status and mental health status were analyzed earlier by various researchers using various methods of analysis. For the purposes of the present exposition, we note here that the data were used earlier to illustrate both the application of association models and the application of correlation models in measuring the observed relationship (the non-independence) between the two polytomous variables in Table 1 (see, e.g., Goodman 1979a, 1985; Gilula and Haberman 1986). These data were also analyzed using a latent class model (see, e.g., Goodman 1987); but in the present exposition we shall examine more fully than earlier how the analysis of latent classes in the present context can change in a dramatic way our view of the observed relationship between the two polytomous variables in Table 1 and in other such tables. We shall also include here some simplification and improvement of some of the results presented in the earlier literature on the analysis of cross-classified data of the kind presented in Table 1 and of the kind presented in the other examples considered herein.

TABLE 1 AROUND HERE

In Table 1, the row categories pertain to parental socioeconomic status (from high parental status to low parental status), and these categories have been numbered (six row categories numbered from 1 to 6); and the column categories pertain to mental health status (from well to impaired), and these categories have also been numbered (four column categories numbered from 1 to 4). These numbers have no special numerical meaning except to indicate which row is being referred to (and possibly where the row appears in the ordering of the rows, if the rows are considered to be ordered) and which column is being referred to (and possibly where the column appears in the ordering of the columns, if the columns are considered to be ordered). With the earlier analysis of Table 1 using correlation models, it was possible to find a set of meaningful numerical scores (different from the integers from 1 to 6) for the row categories and a set of meaningful numerical scores (different from the integers from 1 to 4) for the column categories in Table 1; and the correlation calculated between the meaningful scores for the row categories and the meaningful scores for the column categories for the cross-classified data in Table 1 was also meaningful. The correlation turned out to be small in magnitude, 0.16, but it was statistically significant. Also, with the earlier analysis of Table 1 using the association models, a somewhat similar kind of result was obtained; but, with these models, an index of intrinsic association (rather than an index of correlation) turns out to be meaningful, and the row scores and column scores that are obtained with these association models also turn out to be meaningful (but these scores differ in their meaning from the corresponding scores obtained with the correlation models).

The association models and the correlation models view the two-way 6×4 table in a symmetric way; they consider the association (or the correlation) between the row variable and the column variable, treating the row and column variables symmetrically. On the other hand, these models can also be interpreted in an asymmetrical way in the situation where we might be interested in the possible dependence of, say, the column variable on the row variable. Here we might be interested in, e.g., the possible dependence of mental health status on parental socioeconomic status. With the application of the association models in this context, we can consider the odds of being, say, in mental health status 1 rather than 2, or the odds of being in mental health status 2 rather than 3, or the odds of being in mental health status 3 rather than 4; and we can use the association models to describe how these odds change in a systematic way as we consider these odds for those with different parental socioeconomic status—how these odds change as we move from considering those whose parental socioeconomic status is at the high level to those whose parental socioeconomic status is at a lower level. A somewhat similar kind of result can be obtained when the correlation models are applied to Table 1.

Both the association models and correlation models could be viewed as somewhat improved or somewhat more sophisticated forms of ordinary regression analysis or ordinary correlation analysis, or logit analysis or loglinear analysis. They describe, in one way or another, how the two observed variables, the row variable and the column variable, *appear* to be related to each other, or they describe how one of the variables, say, the column variable, *appears* to be related to the other variable or to be affected by the other variable. For the data in Table 1, the apparent relationship or the apparent effect is statistically significant—nevertheless, I continue to refer here to the apparent (or manifest) relationship or to the apparent (or manifest) effect. All of the methods just mentioned (regression analysis, correlation analysis, logit analysis, loglinear analysis, association model analysis, and correlation model analysis) are concerned with apparent (or manifest) relationships or apparent (or manifest) effects. With the introduction of latent class models, we can examine whether these statistically significant apparent relationships and apparent effects might actually be spurious.

Let us suppose now that there are, say, two kinds of families: One kind I shall call simply the "favorably endowed", and the other kind I shall call the "not favorably endowed". These can be viewed as latent categories or latent classes or latent types of families; and we can consider the latent variable E for "endowment" (favorable endowment or not favorable endowment) as a latent dichotomous variable. And suppose that this latent variable E affects what parental socioeconomic status is attained, and also that it is this latent variable E that affects what is the mental health of the individual. If this is the case, the relationship between variable S (parental socioeconomic status), M (mental health status), and E (endowment status) can be described by Figure 1a, where variables S and M are conditionally independent of each other, given the level of variable E (i.e., given the endowment status of the family). In this case, the statistically significant relationship observed between parental socioeconomic status and mental health status is spurious.

FIG. 1 AROUND HERE

Next let us consider a somewhat different situation. Let us suppose now that there are, say, two kinds of individuals (rather than two kinds of families): One kind I shall call the "favorably endowed", and the other kind I shall call the "not favorably endowed". These can be viewed as latent categories or latent classes or latent types of individuals; and we can consider the latent variable E' for "endowment" (favorable endowment or not favorable endowment) as a latent dichotomous variable. And suppose that it is this latent variable E' that affects what is the mental health status of the individual, and also that it is parental socioeconomic status that affects what is the endowment status E' (favorable or not favorable) of the individual. If this is the case, the relationship between variable S, M, and E' can then be described by Figure 1b, with variable S affecting variable E', and variable E' affecting variable M; and with variables S and M again conditionally independent of each other, given the level of variable E'.

In Figure 1a, the latent variable E is an antecedent variable; and in Figure 1b, the latent variable E' is an intervening variable. In addition to Figures 1a and 1b, we might also consider Figure 1c. Here we again have a somewhat different situation (different from the situations described by Figures 1a and 1b). Figure 1c can be used to describe the situation in which variable S (parental socioeconomic status) and latent variable E (endowment status) reciprocally affect each other (or variables S and E are coincident with each other), and it is variable E that affects what is the mental health status of the individual. (In other contexts, where the column variable M might be viewed as prior to the row variable S, we could consider Figure 1a, and we could also consider the corresponding figures obtained when the symbols S and M are interchanged in Figure 1b and also in Figure 1c.) Each of these figures is congruent with the situation in which variables S and M are conditionally independent of each other, given the level of the latent variable (E or E').[2] Using latent class models, we can examine whether or not this conditional independence is congruent with the data in Table 1.

It may be worthwhile to note here that if variable E (or E') is viewed as antecedent to variables S and M (as in Figure 1a), then we could conclude from Figure 1a that the observed relationship between S and M has been explained away by variable E (or E'); but we could also conclude from Figure 1a that the observed relationship between S and M has been explained (rather than explained away) by variable E (or E')—i.e., by the relationship between variable E (or E') and S and the relationship between variable E (or E') and M. If variable E (or E') is viewed as intervening between variables S and M (as in Figure 1b), or if variable E (or E') is viewed as coincident or reciprocal with variable S and as antecedent to variable M (as in Figure 1c), then we could also conclude from Figure 1b or Figure 1c that the observed relationship between S and M can be explained (rather than explained away) by variable E (or E').

For the cross-classified data in the 6×4 table (Table 1) considered here, we present in Table 2 the chi-square values and the corresponding degrees of freedom that are obtained when these data are analyzed using the following two models: (a) the usual simple null model (H_0) of statistical independence between variables S and M, and (b) the latent class model (H_1) in which the latent variable is dichotomous. (Note that the usual null model H_0 can be described by the simple formula (1) presented earlier herein; and the latent class model H_1 can be described by formula (3) with $T = 2$, and with $I = 6$ and $J = 4$. We could also describe model H_0 by formula (3) with $T = 1$ and $\pi_1^X = 1$.) From the results presented in Table 2, we conclude that (a) the non-independence between variables S and M is statistically significant, and (b) the latent class model H_1 fits the observed data extremely well.

TABLE 2 AROUND HERE

From the results presented in Table 3, we can compare the 24 observed frequencies in the 6×4 table (Table 1) with the corresponding expected values estimated under model H_0 and under model H_1. From these results, we see clearly the dramatic improvement in fit that is obtained when the latent class model H_1 is used as a replacement for the usual null model H_0 of statistical independence between the row variable and the column variable in the two-way table. We also see that the goodness-of-fit chi-square value is reduced by 94 percent (from 45.99 to 2.74) when model H_0 is replaced by model H_1 (with the corresponding reduction in degrees of freedom from 15 to 8). Since model H_1 is the model described in Figures 1a, 1b, and 1c, we find that the cross-classified data in Table 1 are congruent with those figures; and so any of the corresponding interpretations presented earlier herein for these figures can be applied to these data.

TABLE 3 AROUND HERE

The interpretations of the data obtained with Figures 1a, 1b, and 1c are very different in character from the kinds of interpretations obtained earlier when other statistical models (e.g., the association models and/or the correlation models) were used to analyze Table 1. With Figure 1a, we could explain away the statistically significant relationship described earlier with the association and/or correlation models, or we could explain (rather than explain away) this statistically significant relationship. Also with Figure 1b and/or Figure 1c we can explain the statistically significant relationship. With each of these three figures, we can provide a very different kind of explanation of the statistically significant relationship between parental socioeconomic status and mental health status than can be obtained using the association and/or correlation models or any of the other more usual statistical methods of analysis (e.g., loglinear analysis, logit analysis, etc.). We shall comment further on the latent class analysis of Table 1 later herein in the Appendix.

4. A SECOND EXAMPLE. THE ANALYSIS OF THE RELATIONSHIPS AMONG FOUR OBSERVED VARIABLES

As our second example, we consider next the analysis of the cross-classified data presented in Table 4, a four-way table with four observed (manifest) dichotomous variables (say, A, B, C, and D). This table describes the response patterns for respondents to four questionnaire items in which four different situations of role conflict are considered. The respondents are cross-classified in Table 4 with respect to whether they tend toward "*universalistic*" values (+) or "*particularistic*" values (−) when confronted by each of the situations of role conflict.[3] The cross-classified data in Table 4 were analyzed earlier by

Stouffer and Toby (1951) and Lazarsfeld and Henry (1968) using a particular latent class model that had five latent classes, and by Goodman (1974b) using much simpler latent class models that have two latent classes and three latent classes. In the present exposition, we shall examine more fully than earlier the interpretation of the two-class and three-class latent class models in the analysis of the data in Table 4 and in other such tables, and we shall see how the use of such latent class models in the present context can change in a dramatic way our view of the meaning and the character of the underlying latent variable on which our attention is focused. We shall also include in the present section some simplification and improvement of some of the results presented in the earlier literature on the analysis of these data.

TABLE 4 AROUND HERE

For expository purposes, we shall present now the formula for the latent class model in the situation in which there are four observed (or manifest) dichotomous or polytomous variables (say, A, B, C, and D) and one unobserved (or latent) dichotomous or polytomous variable (say, X):

$$\pi_{ijk\ell t}^{ABCDX} = \pi_t^X \pi_{it}^{\bar{A}X} \pi_{jt}^{\bar{B}X} \pi_{kt}^{\bar{C}X} \pi_{\ell t}^{\bar{D}X}, \quad \text{for } i = 1, \ldots, I;\; j = 1, \ldots, J;\; k = 1, \ldots, K;\; \ell = 1, \ldots, L;\; t = 1, \ldots, T, \tag{5}$$

where $\pi_{ijk\ell t}^{ABCDX}$ is the joint probability that an observation is in class i on variable A, in class j on variable B, in class k on variable C, in class ℓ on variable D, and in class t on variable X; and the other terms in formula (5) have the same kind of meaning as the corresponding terms in formula (3). Formula (5) states that variables A, B, C, and D are mutually independent of each other, given the class level on variable X. We shall consider next the following models applied to the cross-classified data in the four-way $2 \times 2 \times 2 \times 2$ table (Table 4): (a) the usual null model (M_0) in which variables A, B, C, and D are mutually independent of each other; (b) the latent class model (M_1) in which the latent variable is dichotomous; and (c) the latent class model (M_2) in which the latent variable is trichotomous. (Note that model M_0 can be described by formula (5) with $T = 1$ and $\pi_1^X = 1$; and the latent class models M_1 and M_2 can be described by formula (5) with $T = 2$ and $T = 3$, respectively. When these models are applied to the four-way table (Table 4), we also have $I = J = K = L = 2$ in formula (5).)

For the cross-classified data in Table 4, we present in Table 5.1 the chi-square values and the corresponding degrees of freedom that are obtained when these data are analyzed using models M_0, M_1, and M_2 described above. From the results presented in Table 5.1, we see that (a) the usual null model M_0, which states that variables A, B, C, and D are mutually independent of each other, definitely does not fit the observed data in Table 4; and (b) the simple latent class model M_1 that has two latent classes fits the observed data very well indeed. When model M_0 is replaced by M_1, there is a dramatic reduction of 97% in the goodness-of-fit chi-square value (from 104.11 to 2.72), with the corresponding reduction in degrees of freedom from 11 to 6.

TABLES 5.1 AND 5.2 AROUND HERE

As we noted earlier, in addition to the results presented in Table 5.1 for the latent class model M_1, which has two latent classes, we also presented there the corresponding results obtained when the latent class model M_2, which has three latent classes, is applied to Table 4. With the three-class latent class model M_2, we shall consider next in Table 5.2 several other latent class models (viz., models M_3, M_4, and M_5) that also have three latent classes. We present now in Table 5.2 the chi-square values and the corresponding degrees of freedom that are obtained when these latent class models are applied to Table 4. Here again we obtain some dramatic results.

Comparing the results presented in Table 5.2 for models M_3, M_4, and M_5 (each of these models having three latent classes) with the corresponding results presented in Table 5.1 for model M_1 (which has two latent classes), we find that models M_3, M_4, and M_5 are *more* parsimonious than model M_1, and that these more parsimonious models fit the observed data essentially as well as the less parsimonious model M_1. (From Tables 5.1 and 5.2, we note that there are 6 degrees of freedom corresponding to model M_1, and 9, 10, and 11 degrees of freedom corresponding to M_3, M_4, and M_5, respectively; and the goodness-of-fit chi-square values range from 2.28 for M_3 to 2.85 for M_5, with a chi-square value of 2.72 for M_1.) Indeed, again comparing corresponding results in Tables 5.1 and 5.2, we see that the latent class model M_5 in Table 5.2 turns out to be as parsimonious as the simple null model M_0 of mutual independence among the four observed variables (A, B, C, and D), with 11 degrees of freedom corresponding to model M_0 and also to model M_5; and the goodness-of-fit chi-square value obtained under M_5 is also 97% less than the corresponding chi-square value obtained under M_0.

Figure 2 can be used to describe any of the latent class models (M_1 to M_5) in Tables 5.1 and 5.2, where the latent variable (dichotomous or trichotomous) is represented as variable V for "values" ("universalistic" values or "particularistic" values). In this figure, the latent variable V is viewed as an antecedent variable that can explain away the observed relationships among the observed (manifest) variables A, B, C, and D, or that can explain (rather than explain away) these observed relationships among the observed variables. We can also view the latent variable V as the variable of interest, and the observed variables A, B, C, and D as indicators or markers for the latent variable. Applying the latent class models to the observed data in Table 4, using the observed variables as indicators or markers, we are able to describe and measure the unobserved latent variable (i.e., the "true" latent types). We present the description of this latent variable next for models M_1 and M_3 in Tables 6 and 7.

FIG. 2 AROUNI HERE

In model M_1, there are two latent classes, a "universalistically inclined" latent class and a "particularistically inclined" latent class. From Table 6, we see that the probability of a universalistic response in situation A is .993 for those who are in the universalistically inclined latent class; and for situations B, C, and D, the corresponding probabilities are .940, .927, and .769. Also, the probability of a universalistic response in situation A is .714 for those who are in the particularistically inclined latent class; and for situations B, C, and D, the corresponding probabilities are .330, .354, and .132. The modal response is universalistic in situations A, B, C, and D for those who are in the universalistically inclined latent class; and the modal response is particularistic in situations B, C, and D for those who are in the particularistically inclined latent class. (In situation A, the modal response is universalistic both for those who are in the universalistically inclined latent class and for those who are in the particularistically inclined latent class; but the corresponding probability of a universalistic response is reduced in the latter latent class from .993 to .714.) If we think of the responses in items A, B, C, and D as indicators, measured with error, of the unobserved latent class (the "true" latent type), then the error rates for items A, B, C, and D are .007, .060, .073, and .231, respectively, for those who are in the universalistically inclined latent class; and the error rates are .714, .330, .354, and .132, respectively, for those who are in the particularistically inclined latent class. The proportion of individuals who are in the universalistically inclined latent class is estimated to be .28, and the proportion in the particularistically inclined latent class is estimated to be .72.

TABLE 6 AROUND HERE

Now for model M_3 in Table 7. For this model, there are three different latent classes (rather than two); and they are the "strictly universalistic", the "mixed universalistically/ particularistically inclined", and the "strictly particularistic". The "strictly universalistic" always respond universalistically in situations A, B, C, and D; and the "strictly particularistic" always respond particularistically in situations A, B, C, and D. And for those in the middle latent class, the "mixed universalistically/particularistically inclined", we see that the probability of a universalistic response varies from .796 in situation A to .175 in situation D. The proportion of individuals who are in each of the three latent classes is estimated to be .17, .78, and .05 for the strictly universalistic, and the mixed universalistically/particularistically inclined, and the strictly particularistic, respectively.

TABLE 7 AROUND HERE

Models M_1 and M_3 make different statements about the heterogeneity of the individuals whose responses in situations of role conflict are described in Table 4. Model M_1 states that there are two different types of individuals, and the model can be used to describe how these two types differ from each other; and model M_3 states that there are three different types of individuals, and the model can be used to describe how these three types differ from each other. Each of the latent class models considered herein can be viewed as making a statement about the heterogeneity of the individuals under investigation.

From the information given in Tables 6 and 7, we can calculate, under model M_1 and under model M_3, the expected number of respondents who will be in each of the $2 \times 2 \times 2 \times 2 = 16$ possible cells of the four-way table who are from each of the latent classes; and, for each of the 16 cells in the four-way table, we can also calculate the chance that a respondent in that cell is from each of the latent classes. (Under model M_1, each of the two latent classes—the "universalistically inclined" class and the "particularistically inclined" class—contributes to the expected number of respondents who will be in each of the 16 cells of the four-way table; whereas, under model M_3, only the "mixed universalistically/particularistically inclined" latent class contributes to the expected number of respondents who will be in each of the 16 cells of the four-way table, the "strictly universalistic" latent class contributes to the expected number of respondents only in the extreme (+,+,+,+) cell, and the "strictly particularistic" latent class contributes to the expected number of respondents only in the extreme (−,−,−,−) cell; see, e.g., Tables 4, 6, and 7.)

Models M_1 and M_3 are very different from each other. The corresponding latent variables (see Tables 6 and 7) are also very different from each other (with the "universalistically inclined" latent class and the "particularistically inclined" class in model M_1; and the "strictly universalistic" latent class, the "strictly particularistic" latent class, and the "mixed universally/particularistically inclined" class in model M_3). While there are two latent classes in model M_1 and three latent classes in model M_3, we can view model M_3 as consisting "essentially" of only one latent class (the "mixed universalistically/particularistically inclined" latent class) contributing to each of the cells in the four-way table, with a special addition applied to the extreme (+,+,+,+) cell (contributed by the "strictly universalistic" latent class) and a special addition applied to the extreme (−,−,−,−) cell (contributed by the "strictly particularistic" latent class). As we noted earlier from Tables 5.1 and 5.2, model M_3 is more parsimonious than model M_1, even though M_3 includes one more latent class than does M_1. We also noted from Tables 5.1 and 5.2 that both of these models fit the data very well indeed; and we also see from these tables that model M_3 fits the data slightly better than does M_1.

Models M_4 and M_5 in Table 5.2 are very similar to model M_3, with one added constraint (viz., $\pi_{12}^{\bar{B}V} = \pi_{12}^{\bar{C}V}$) in model M_4, and two added constraints (viz., $\pi_{12}^{\bar{B}V} = \pi_{12}^{\bar{C}V}$ and $\pi_{12}^{\bar{A}V} = \pi_{22}^{\bar{D}V}$) in model M_5. We present the results obtained with model M_5 in Table 8 (see also, e.g., Goodman 1979b). As we noted earlier herein, model M_5 is more parsimonious than any of the other latent class models (the two-class and three-class latent class models) in Tables 5.1 and 5.2, and it is as parsimonious as the simple null model M_0 of mutual independence. (And, as we also noted earlier herein, model M_5 also fits the observed data very well indeed—with a 97% improvement in fit when the fit of this model is compared with that of the simple null model M_0.) Compare the distribution of responses in Table 8 for situations B and C, and also for situations A and D. We have here in Table 8 an explanation (or description) of the data in Table 4 that is very parsimonious indeed. TABLE 8 AROUND HERE

We have commented in this section on each of the models in Tables 5.1 and 5.2 except for model M_2. This model will be considered later herein in the Appendix.

5. ANOTHER EXAMPLE. THE SIMPLE 2×2 CROSS-CLASSIFICATION TABLE REVISITED: A NINETEENTH CENTURY LATENT STRUCTURE AND SOME NEW ALTERNATIVES

We return now to the analysis of the relationship between two observed variables in the simple context of the 2×2 cross-classification table $\{A, B\}$, where the observed variables A and B are both dichotomous. In the introductory section of this exposition, we began with this topic; and, in a comment in that section that pertained to nineteenth-century work on this topic, we included there a reference to the work of C. S. Peirce (1884) on the analysis of the 2×2 table in the context of measuring the success of predictions. We shall now show how Peirce's work can be interpreted as an example of the early use of a latent class model in the analysis of a 2×2 table; and we shall also show how the introduction of latent class models in this context can lead to some additional simple measures that are different from the one proposed by Peirce.

With respect to the prediction of, say, tornadoes, where the dichotomous row variable A pertains to tornado prediction (predict "tornado" or predict "no tornado"), and the dichotomous column variable B pertains to tornado occurrence (tornado occurs or no tornado occurs), Peirce pointed out that the observed results in the 2×2 cross-classification table could be viewed as having been obtained by using an infallible predictor a proportion θ of the time, and a completely ignorant predictor the remaining proportion $1 - \theta$ of the time. The infallible predictor predicts "tornado" if a tornado will occur, and he/she predicts "no tornado" if a tornado will not occur; and, for the ignorant predictor, the chance that he/she will predict "tornado" or "no tornado" is independent of whether the tornado will occur or will not occur. We are asked to contemplate a mixture of the following two 2×2 sets of probabilities, pertaining to the infallible predictor and to the ignorant predictor, respectively:

		B: 1	B: 2
A	1	P_1^B	0
	2	0	P_2^B

		B: 1	B: 2
A	1	$\alpha_1 P_1^B$	$\alpha_1 P_2^B$
	2	$\alpha_2 P_1^B$	$\alpha_2 P_2^B$

with weights θ and $1-\theta$, respectively, where the meaning of the four cells is the same as in the 2×2 table $\{A, B\}$ for the dichotomous row variable A and the dichotomous column variable B defined earlier in this paragraph; and where P_1^B and P_2^B are the probability of tornado occurrence and the probability of no tornado occurrence, respectively; and where α_1 and α_2 are the probability of the "tornado" prediction and the probability of the "no tornado" prediction, respectively, for the ignorant predictor. Using this model, Peirce found that the proportion θ of the time that an infallible predictor is used can be determined by the following formula:

$$\theta = \frac{P_{11}P_{22} - P_{12}P_{21}}{P_1^B P_2^B} = \frac{P_{11}}{P_1^B} - \frac{P_{12}}{P_2^B}; \tag{6}$$

see Peirce (1884) and Goodman and Kruskal (1959).

Peirce's model can be viewed as a latent structure in which the observed cross-classification in the 2×2 table $\{A, B\}$ is obtained from a mixture of the two 2×2 tables, which we described above, pertaining to an unobserved (or latent) infallible predictor and an unobserved (or latent) ignorant predictor, respectively, where the proportion of the time that each kind of latent predictor is used is unknown. This latent structure is *not*, strictly speaking, a latent class model, because the first 2×2 table (i.e., the table pertaining to the infallible predictor) does *not* pertain to a latent class in which there is independence between the dichotomous row variable A and the dichotomous column variable B. We shall next introduce a latent structure that is more general than Peirce's model and that is a latent class model.

Let us consider a three-class latent class model for the 2×2 table $\{A, B\}$, in which the first two latent classes pertain to two different kinds of infallible predictors, and the third latent class pertains to an ignorant predictor, with

$$\pi_{11}^{\bar{A}X} = \pi_{11}^{\bar{B}X} = 1, \tag{7.1}$$

and

$$\pi_{22}^{\bar{A}X} = \pi_{22}^{\bar{B}X} = 1. \tag{7.2}$$

Peirce's model can be viewed as a special case of this three-class latent class model in which the following additional constraints are imposed on the parameters in the model:

$$\pi_{13}^{\bar{B}X} = P_1^B, \quad \pi_{23}^{\bar{B}X} = P_2^B, \tag{8.1}$$

and

$$\pi_1^X = \theta P_1^B, \quad \pi_2^X = \theta P_2^B, \tag{8.2}$$

where the value of θ is unknown. (Note that $\theta = \pi_1^X + \pi_2^X$ in this special case.) With the imposition of the additional constraints, (8.1) and (8.2), in the above three-class latent class model, we can then obtain formula (6) for θ.

Next let us briefly introduce two other special cases of the above three-class latent class model (when (7.1)–(7.2) holds true). Instead of imposing the additional constraints (8.1) and (8.2) as in Peirce's model, let us consider, for example, the model obtained with the imposition of the simple constraint $\pi_2^X = 0$ in the three-class latent class model. With this model, there is just one kind of infallible predictor (rather than the two kinds) in the model, and the proportion θ' of the time that the infallible predictor is used can be determined simply by

$$\theta' = P_{11} - P_{12}P_{21}/P_{22}, \tag{9}$$

with $\theta' = \pi_1^X$. Similarly, with the imposition of the constraint $\pi_1^X = 0$ (rather than the constraint $\pi_2^X = 0$) in the three-class latent class model, it is the other kind of infallible predictor that is in the model, and the corresponding proportion θ'' of the time that this infallible predictor is used can be determined simply by

$$\theta'' = P_{22} - P_{12}P_{21}/P_{11}, \tag{10}$$

with $\theta'' = \pi_2^X$.

We have presented above three simple but different formulae (viz., (6), (9), and (10)) for determining the proportion of time that use is made of the unobserved (latent) infallible predictor (or predictors). The three formulae, (6), (9), and (10), define the corresponding three measures, θ, θ', and θ'', respectively; and the relationships among these three measures can be described simply as follows:

$$\theta = \theta' P_{22}/(P_1^B P_2^B) = \theta'' P_{11}/(P_1^B P_2^B), \tag{11.1}$$

and

$$\theta'/\theta'' = P_{11}/P_{22}. \tag{11.2}$$

In some substantive contexts, the three measures can yield very different results. The three latent class models that yield the three measures are quite different from each other (and the three sets of constraints pertaining to the three models—viz., (8.1)–(8.2), and $\pi_2^X = 0$, and $\pi_1^X = 0$—are quite different from each other). Which of the three measures is appropriate will depend upon the substantive context (i.e., on which of the three sets of constraints pertaining to the three latent class models is compatible with the substantive context).

The above latent class models treat the cells on the main diagonal in the 2×2 cross-classification table in a way that is different from the way that the other cells in the table are treated (with the latent infallible predictor or predictors directly affecting the entries in the cells on the main diagonal and not directly affecting the entries in the other cells in the table). The need for special treatment for the cells on the main diagonal arises in various substantive contexts in the analysis of 2×2 tables and in the analysis of other square $I \times I$ cross-classification tables (i.e., $I \times J$ tables with $J = I$), where there is a one-to-one correspondence between the I row categories and the I column categories. (In the 2×2 table $\{A, B\}$ considered earlier in this section, the row variable A pertained to whether the prediction was that an event would occur or that it would not occur, and the column variable B pertained to whether the event then did occur or did not occur.) For the analysis of the 2×2 table $\{A, B\}$, we introduced in this section the three-class latent class model with the conditional probabilities pertaining to the first two latent classes (viz., $\pi_{i1}^{\bar{A}X}$, $\pi_{i2}^{\bar{A}X}$, $\pi_{j1}^{\bar{B}X}$, $\pi_{j2}^{\bar{B}X}$) restricted in a way so that these two latent classes would directly affect only the entries in the two cells on the main diagonal in the 2×2 table (see (7.1)–(7.2)); and for the analysis of the $I \times I$ table (with $I > 2$), we can consider the corresponding $(I+1)$-class latent class model with corresponding restrictions imposed on the first I latent classes (so that these I latent classes would directly affect only the entries in the corresponding I cells on the main diagonal in the $I \times I$ table). The parameters $\pi_1^X, \pi_2^X, \ldots, \pi_I^X$ in the $(I+1)$-class latent class model can then be estimated using the following formula:

$$\pi_t^X = P_{tt} - \pi_{t,t,I+1}^{ABX}, \text{ for } t = 1, 2, \ldots, I, \tag{12}$$

with $\pi_t^X \geq 0$, and where P_{tt} is the probability that an observation will fall in the t-th row and t-th column of the $I \times I$ table (i.e., in the t-th cell on the main diagonal, for $t = 1, 2, \ldots, I$), and $\pi_{t,t,I+1}^{ABX}$ is the corresponding probability, under the latent class model, that an observation will fall in the t-th row and t-th column of the $I \times I$ table *and* in the $(I+1)$-th latent class of the latent class model. (Note that $P_{tt} \geq \pi_{t,t,I+1}^{ABX}$ in this model.) The formula for $\pi_{t,t,I+1}^{ABX}$ in (12) is obtained from formula (3) for the latent class model, with

$$\pi_{t,t,I+1}^{ABX} = \pi_{I+1}^{X} \pi_{t,I+1}^{\bar{A}X} \pi_{t,I+1}^{\bar{B}X}. \tag{13}$$

(Compare formula (13) with formula (3); and also compare formula (12) with formulae (9) and (10).) The probability P_{tt} in formula (12) is estimated simply by the observed proportion p_{tt} of observations in the t-th row and t-th column of the $I \times I$ table; and the estimate $\hat{\pi}_{t,t,I+1}^{ABX}$ of the corresponding $\pi_{t,t,I+1}^{ABX}$ in formula (12) can be obtained from formula (13) using the model of "quasi-independence" applied to the observed cross-classification in the $I \times I$ table (when $I > 2$) with the entries deleted in the cells on the main diagonal in this table (see, e.g., Goodman 1968, 1969a, 1969b).[4]

The preceding paragraph considered the analysis of the 2×2 table and the more general square $I \times I$ table (for $I \geq 2$) in the situation on which there is a need for special treatment for the I cells on the main diagonal of the table. (For this situation, we considered the $(I+1)$-class latent class model described above, and the corresponding quasi-independence model applied with the entries deleted in the I cells on the main diagonal (when $I > 2$).) Similarly, we could also consider the two-way 2×2 table and the more general multi-way table (i.e., the m-way table, for $m \geq 2$) in situations in which there is a need for special treatment for a specified subset of the cells in the table. Situations of this kind arise in various substantive contexts; and the method of analysis presented in the preceding paragraph can be directly extended to such situations. For example, in the situation in which there is a need for special treatment for, say, S specified cells in the m-way table, we could consider the $(S+1)$-class latent class model with appropriate restrictions (of the kind described above) imposed on the first S latent classes (which correspond to the S specified cells in which there is a need for special treatment—see, e.g., (7.1)–(7.2) for $S = 2$ and $m = 2$), and the corresponding m-way quasi-independence model applied with the entries deleted in the S specified cells of the m-way table. This general approach was applied, e.g., in order to introduce a new latent class model for scaling response patterns in the analysis of, say, an m-way 2^m table (for $m > 2$) in which there is a need for special treatment for $m+1$ specified cells in the table—viz., the cells that correspond to the $m+1$ scalable types in the table (see, e.g., Goodman, 1975). In this special case, we have $S = m+1$, and the corresponding latent class model has $S+1 = m+2$ latent classes; and, with the application of the corresponding m-way quasi-independence model, the entries are deleted in the cells that correspond to the $m+1$ scalable types.[5]

6. SOME NOTES ON THE HISTORY OF LATENT CLASS ANALYSIS

In the introductory section of this exposition, we noted that, in contrast to the problem of measuring the relationship (the non-independence) between two (or more) observed dichotomous or polytomous variables, which many researchers have considered at various times throughout the twentieth century and also at times during the nineteenth century,[6] the main development of latent class models has taken place only during the last half of this past century.[7] The literature on this topic that appeared in the 1950's and the 1960's was limited primarily to the situation in which all of the observed (manifest) variables were dichotomous (not polytomous). Lazarsfeld first introduced the term "latent structure" models in 1950; and the various models that he considered as latent structure models (including the latent class model) were all concerned with "dichotomous systems" of observed variables. During the 1950's and 1960's, there were essentially five different methods that were proposed for estimating the parameters in the latent class model; viz., (1) a method suggested by Green (1951) that resembled in some respects traditional factor analysis; (2) a method suggested by Gibson (1951), which was quite different from the method suggested by Green (1951), but which also resembled factor analysis in some respects;[8] (3) a method that was based on the calculation of the solution of certain determinantal equations, which was suggested by the work of Lazarsfeld and Dudman (1951) and Koopmans (1951), and was developed by Anderson (1954a), and extended by Gibson (1955) and Madansky (1960); (4) a scoring method described by McHugh (1956) for obtaining maximum-likelihood estimates of the parameters in the model; and (5) a partitioning method developed by Madansky (1959) that is based on an examination of each of the possible assignments of the observations in the cross-classification table to the different latent classes. It turned out that (1) the first method (Green's method), or a version of this method, can provide estimators of the parameters in the latent class model that are not consistent (see, e.g., Madansky 1968);[9] (2) the second method (Gibson's 1951 method) also can provide estimators of the parameters in the latent class model that are not consistent;[10] (3) the third method (the determinantal equations method) does provide (under certain conditions) consistent estimators, but these estimators are not efficient, and some of the estimates that are actually obtained using this method are often not permissible (i.e., not admissible) as estimates of the corresponding parameters in the latent class model (see, e.g., Anderson 1959; Anderson and Carleton 1957);[11] (4) the fourth method (the scoring method) can provide (under certain conditions) efficient estimators, but the estimates that are actually obtained with this method, as described in McHugh

(1956, 1958), can have a similar kind of permissibility problem associated with it as did the determinantal equations method, and an implementation of this procedure in the early 1960's was judged too costly for practical use (see, e.g., Madansky 1968; Henry 1983); and (5) the fifth method (the partitioning method) has certain merits (e.g., one version of this method can provide efficient estimators), but the shortcoming of this method is that it is too time consuming, even with fast computers and samples that are not very large, to enumerate and assess each of the possible assignments of the observations in the cross-classification table to the different latent classes (Madansky 1968).

In assessing what was the state of affairs near the end of the 1960's with respect to the estimation of the parameters in the latent class model, Lazarsfeld and Henry (1968) comment as follows: "A great deal of imaginative thinking and sophisticated programming is still needed before latent class analysis can be routinely applied to a set of data". This state of affairs changed in the middle of the 1970's.

With the introduction then of more general latent class models and the introduction of a relatively simple method for obtaining the maximum-likelihood estimates of the parameters in these models, the practical application of these models by researchers in various fields of inquiry became a realistic possibility (see Goodman 1974a, 1974b).[12] In contrast to the earlier latent class models that were limited to the analysis of only dichotomous (not polytomous) manifest variables and that included only one latent variable in the model, the new more general latent class models referred to above have the following advantages: (1) They can be applied to both dichotomous and polytomous observed (manifest) variables; (2) they can include one or more than one unobserved (latent) variable or variables in the model; and (3) they can include a wide range of possible constraints that can be imposed (if desired) on the parameters in the model.

With the wide range of possible constraints that can be imposed (if desired) on the parameters in the more general latent class model, we can obtain a wide range of useful models. Models that can include properly specified constraints can enable the researcher to test a wide range of hypotheses about the structure of the data. With the more general latent class model and with the many possible models that can be obtained when different constraints are specified, we obtain models that can be applied in many different contexts. For example, the more general latent class model can be used to obtain various models for analyzing the scalability of response patterns (viz., the scaling models—see, e.g., Goodman 1975; Clogg and Sawyer 1981), various models that can include error-rate parameters (measurement error, response error, or classification error parameters) to describe how each of the different latent classes (or some of these latent classes) in the model may respond (with error) in the corresponding observed manifest variables (viz., the measurement models—see, e.g., Goodman 1974a; Clogg and Sawyer 1981), various models that allow for the special treatment of the entries in specified cells of the cross-classification table (viz., the quasi-latent-structure models—see, e.g., Goodman 1974a; Clogg 1981a), various models that include both multiple observed indicator variables and multiple observed antecedent (exogenous or causal) variables (viz., the multiple-indicator-multiple-cause (MIMIC) models—see, e.g., Goodman 1974a, Sec. 5.4; Clogg 1981b), and various models that can be used for the simultaneous analysis and comparison of the latent structures pertaining to the cross-classified data in two or more multi-way tables (viz., the simultaneous latent structure models—see, e.g., Clogg and Goodman 1984, 1985, 1986; and Birkelund, Goodman, and Rose 1996).[13]

With the estimation method introduced in Goodman (1974a,1974b) for obtaining maximum-likelihood estimates for the more general latent class models, we obtained a method of estimation that is quite different from the five estimation methods (considered earlier in this section) that had been developed earlier specifically for latent class models. The relatively simple method introduced in Goodman (1974a,1974b) can be viewed simply as a direct extension of (1) the iterative proportional fitting method (fitting one-way margins) used with the quasi-independence model in the analysis of a two-way table (or a multi-way table) in which the usual independence model is of interest but some of the entries in the table are missing and/or there is a need for special treatment for some of the cells in the table (see, e.g., Goodman 1968), and (2) the iterative proportional fitting method (fitting specified one-way marginals, specified two-way marginals, etc.) used with the usual hierarchical loglinear models in the analysis of a multi-way table pertaining to the joint distribution of a set of manifest variables (see, e.g., Goodman 1970). The iterative proportional fitting method for a quasi-independence model can be used to take into account the fact that there are missing (unobserved) entries in the two-way table (or the multi-way table) pertaining to the joint distribution of two (or more) manifest variables; and the iterative proportional fitting method introduced in Goodman (1974a,1974b) can be used to take into account the fact that there are missing (unobserved) variables (viz., the latent variables) in the multi-way table pertaining to the joint distribution of the manifest variables and the latent variables. With the iterative proportional fitting method for the analysis of a multi-way table pertaining to the joint distribution of a set of manifest variables directly extended to obtain the iterative proportional fitting method for the analysis of a multi-way table pertaining to the joint distribution of a set of manifest variables and latent variables, we obtained an estimation method that can be applied in many contexts in which a set of manifest variables and a related set of latent variables are of interest and also in many contexts in which a set of latent variables and a related set of manifest variables are of interest.

The analysis of a multi-way table pertaining to the joint distribution of a set of manifest variables was extended earlier in order to obtain models for the simultaneous analysis and comparison of two or more such multi-way tables (see, e.g., Goodman 1973a), and it was also extended earlier in order to obtain models for the situation in which some of the manifest variables in the multi-way table are posterior to other manifest variables in the table, where the manifest variables can be viewed as ordered from first to last, or where the set of manifest variables can be partitioned into mutually exclusive and exhaustive subsets which can then be viewed as ordered from first to last (see, e.g., Goodman 1973b and 1973c, and the corresponding path diagrams presented in these articles). It may also be worth noting here that the analysis of the multi-way table in Goodman (1970) listed all possible elementary loglinear models (for the m-way table, with $m = 2, 3, 4$) that can be described in terms of the concepts of independence and conditional independence; and these models would include, for example, the Markov-type models.[14] The models referred to in the present paragraph for the analysis of a multi-way table pertaining to the joint distribution of a set of manifest variables (or for the analysis of two or more such multi-way tables) can be extended now to the analysis of the relationship between a set of manifest variables and a corresponding set of latent variables. See, e.g., Goodman (1974a, Sec. 5.4), Hagenaars (1988,1990,1993), van de Pol and Langeheine (1990), and Vermunt (1997).

Reference should also be made, of course, to many other publications by others, in the 1970's and later, that also contribute to the further development of this subject. With respect to, say, the estimation procedures for latent class models, the reader is referred to, e.g., Haberman (1976,1977,1988), Dempster, Laird, and Rubin (1977), and Vermunt (1997,1999).[15] In addition to the reference to Haberman's work (on estimation procedures for latent class models) cited above, we also refer the interested reader to Haberman (1974, 1979) for related material. With respect to, say, the various reviews of the latent class models literature, which covered work done in the late 1970's and later, we cite here, e.g., Clogg (1981b), Clogg and Sawyer (1981), Andersen (1982), Bergan (1983), McCutcheon (1987), Clogg (1988), Langeheine (1988), Clogg (1995). The reader is referred to these reviews for additional references. Many other contributions to this field could also be cited; but this would be beyond the scope of this exposition on latent class analysis and these brief notes on its history. We are now pleased to be able to refer the reader to the chapters that follow in this book on latent class analysis.

APPENDIX

In this Appendix, we shall comment further on the latent class analysis presented earlier herein of the cross-classified data in the two-way 6×4 table (Table 1) and the cross-classified data in the four-way $2 \times 2 \times 2 \times 2$ table (Table 4). For the analysis of the data in the 6×4 table, the latent class model that was used had two latent classes (i.e., the latent variable was dichotomous). With the formula for this model (see formula (3) earlier herein) we can determine how many basic parameters are included in this model. Replacing the symbols A, B, and X in formula (3) by the corresponding S (parental socioeconomic status), M (mental health status) and E (endowment status), we see that the basic parameters in the model are π_1^E (with $\pi_2^E = 1 - \pi_1^E$), and

$$\pi_{1t}^{\bar{S}E}, \pi_{2t}^{\bar{S}E}, \ldots, \pi_{5t}^{\bar{S}E} \text{ (with } \pi_{6t}^{\bar{S}E} = 1 - \sum_{i=1}^{5} \pi_{it}^{\bar{S}E}), \text{ for } t = 1, 2,$$

and

$$\pi_{1t}^{\bar{M}E}, \pi_{2t}^{\bar{M}E}, \pi_{3t}^{\bar{M}E} \text{ (with } \pi_{4t}^{\bar{M}E} = 1 - \sum_{j=1}^{3} \pi_{jt}^{\bar{M}E}), \text{ for } t = 1, 2.$$

Thus, we see that there are $1 + 5(2) + 3(2) = 17$ basic parameters in the model. It is possible to estimate the expected frequencies under this model using the cross-classified data in the 6×4 table (see the corresponding estimates in Table 3) without actually estimating the basic parameters in the model (see, e.g., Goodman 1974b); but, if these parameters are of substantive interest, then we need to note that they become identifiable only after two of these parameters are specified. Any two of these parameters can be specified (in any way that does not constrain the expected frequencies estimated under the model), and the other 15 parameters then become identifiable, and they can be estimated using efficient statistical methods.

We present in Table A.1 two different sets of numerical values that are obtained as estimates for the parameters in the latent class model after two of the parameters in the model are specified in different ways. The two different sets of estimates in Table A.1 yield the same estimated expected frequencies presented in Table 3 for the latent class model H_1. The first set of estimates in Table A.1 describe a model (say, model H_1') in which there is a stringent threshold for the favorably endowed (with $\pi_{61}^{\bar{S}E} = 0$ and $\pi_{41}^{\bar{M}E} = 0$; i.e., where the conditional probability is zero of being in the lowest parental socioeconomic class for those who are in the favorably endowed latent class, and the conditional probability is zero of being in the worst mental health class for those who are in the favorably endowed class); and, similarly, the second set of estimates in Table A.1 describe a model (say, model H_1'') in which there is a stringent threshold for the not favorably endowed (with $\pi_{12}^{\bar{M}E} = 0$ and $\pi_{22}^{\bar{S}E} = 0$; i.e., where the conditional probability is zero of being in the best mental health class for those who are in the not favorably endowed latent class, and the conditional probability is zero of being in the second highest parental socioeconomic class for those who are in the not favorably endowed class). In the first model (H_1'), 30% are estimated to be in the favorably endowed latent class; and in the second model (H_1''), 23% are estimated to be in the not favorably endowed latent class. As we have already noted, these two models (H_1' and H_1'') are equivalent in the sense that they yield the same estimated expected frequencies (see Table 3). The two equivalent models can be viewed as different versions of model H_1. The estimate of the proportion favorably endowed (i.e., π_1^E) under model H_1 can not be less than .30, and the estimate of the proportion not favorably endowed (i.e., π_2^E) under model H_1 can not be less than .23.

TABLE A.1 AROUND HERE

We present in Table A.2 another two different sets of numerical values that are obtained as estimates for the parameters in the latent class model after two of the parameters in the model are again specified in different ways. Again, the two different sets of estimates in Table A.2 yield the same estimated expected frequencies presented in Table 3 for the latent class model H_1. The first set of estimates in Table A.2 describe a model (say, model H_1''') in which the difference is maximized between the conditional distribution of parental socioeconomic status for the favorably endowed and the corresponding conditional distribution for the not favorably endowed (with $\pi_{61}^{\bar{S}E} = 0$ and $\pi_{22}^{\bar{S}E} = 0$); and similarly, the second set of estimates in Table A.2 describe a model (say, model H_1'''') in which the difference is maximized between the conditional distribution of mental health status for the favorably endowed and the corresponding conditional distribution for the not favorably endowed (with $\pi_{41}^{\bar{M}E} = 0$ and $\pi_{12}^{\bar{M}E} = 0$). In the first model (H_1''') in Table A.2, 61% are estimated to be in the favorably endowed latent class and 39% in the not favorably endowed class; and in the second model (H_1''''), 48% are estimated to be in the favorably endowed latent class and 52% in the not favorably endowed class.

TABLE A.2 AROUND HERE

Each latent class model in Tables A.1 and A.2 is described there in terms of the estimates for the four conditional distributions ($\pi_{i1}^{\bar{S}E}, \pi_{i2}^{\bar{S}E}, \pi_{j1}^{\bar{M}E}, \pi_{j2}^{\bar{M}E}$) and the corresponding estimate of the distribution (π_1^E, π_2^E) of the latent classes. Comparing the estimates for models H_1' and H_1'' in Table A.1, we see the extent to which the conditional distributions can change as the distribution of the latent classes goes from one extreme (when there is a stringent threshold for the favorably endowed) to the other extreme (when there is a stringent threshold for the not favorably endowed); and comparing the estimates for models H_1''' and H_1'''' in Table A.2, we see the extent to which the conditional distributions can change as the differences in the conditional distributions ($|\pi_{i1}^{\bar{S}E} - \pi_{i2}^{\bar{S}E}|$ and $|\pi_{j1}^{\bar{M}E} - \pi_{j2}^{\bar{M}E}|$) go from one extreme (when $|\pi_{i1}^{\bar{S}E} - \pi_{i2}^{\bar{S}E}|$ is maximized) to the other extreme (when $|\pi_{j1}^{\bar{M}E} - \pi_{j2}^{\bar{M}E}|$ is maximized).

As we noted earlier in this Appendix, the 17 basic parameters in the latent class model for the 6×4 table (see formula (3) earlier herein) become identifiable and can be estimated after two of these parameters are specified. Tables A.1 and A.2 present four different examples of how two of the parameters in each example can be specified. The basic parameters can also be estimated by specifying the numerical value of the parameter π_1^E and the numerical value of a second parameter that I shall call the σ multiplier. The parameter σ can be introduced into the latent class model in a way that can increase or decrease the difference $(\pi_{i1}^{\bar{S}E} - \pi_{i2}^{\bar{S}E})$ and can thus decrease or increase the corresponding difference $(\pi_{j1}^{\bar{M}E} - \pi_{j2}^{\bar{M}E})$. As we noted earlier in this Appendix, for the data in Table 1 the π_1^E parameter needs to be within the closed range from .30 to .77, and the σ parameter is also limited in its range. (For further details, see, e.g., formulae in $(A.9)$–$(A.10)$ and footnote 23 later herein.)

Before closing this part of the Appendix on the estimation of the parameters in the latent class model for the data in Table 1, it is interesting to note from Tables A.1 and A.2 that the estimates of $\pi_{i1}^{\bar{S}E}$ are the same under H_1' and H_1''' and under H_1'' and H_1'''' (and this is true also for the estimates of $\pi_{j2}^{\bar{M}E}$); and the estimates of $\pi_{j1}^{\bar{M}E}$ are the same under H_1' and H_1'''' and under H_1'' and H_1''' (and this is true also for the estimates of $\pi_{i2}^{\bar{S}E}$). The parameter $\pi_{61}^{\bar{S}E}$ was specified in the same way under H_1' and H_1''' (but it was not specified under H_1'' and H_1''''); the parameter $\pi_{41}^{\bar{M}E}$ was specified in the same way under H_1' and H_1'''' (but it was not specified under H_1'' and H_1'''); the parameter $\pi_{22}^{\bar{S}E}$ was specified in the same way under H_1'' and H_1''' (but it was not specified under H_1' and H_1''''); and the parameter $\pi_{12}^{\bar{M}E}$ was specified in the same way under H_1'' and H_1'''' (but it was not specified under H_1' and H_1''').

The description of the two-class latent class models in Tables A.1 and A.2 (viz., models

$H_1', H_1'', H_1''', H_1'''')$ applied to the cross-classified data in the 6×4 table (Table 1) viewed the two-way table in a symmetric way in the sense that the models could be described by Figure 1a and by formula (3) presented earlier herein (with A, B, and X in formula (3) replaced by S, M, and E, respectively, in Figure 1a).[16] These tables presented the distribution of parental socioeconomic status and the distribution of mental health status for the favorably endowed latent class and the not favorably endowed latent class. As we noted earlier, in addition to the symmetric view of the two-way table presented in Figure 1a, we also can view the two-way table in an asymmetric way with Figures 1b and 1c. With the asymmetric view presented in Figure 1b, our interest is focussed on the distribution of the favorably endowed and the not favorably endowed (i.e., the "endowment" distribution) for those in each parental socioeconomic status category (rather than on the distribution of parental socioeconomic status for those in the favorably endowed latent class and for those in the not favorably endowed class); see footnote 2 earlier herein. We consider next Table A.3, which presents the endowment distribution for those in each parental socioeconomic status category, and also the corresponding endowment distribution for those in each mental health status category,[17] under the latent class models H' and H'' (which we considered earlier in Table A.1).[18]

TABLE A.3 AROUND HERE

As we noted earlier from Table A.1, for the favorably endowed latent class, the estimate of π_1^E, under models H' and H'', was .30 and .77, respectively; and thus, for the not favorably endowed latent class, the estimate of π_2^E under H' and H'' was .70 and .23, respectively. (Under model H', there was a stringent threshold for the favorably endowed; and under model H'', there was a stringent threshold for the not favorably endowed.) The difference between the endowment distributions under model H' and the corresponding endowment distributions under model H'' reflects, to a large extent, the difference between the estimate of π_1^E under H' and the corresponding estimate of π_1^E under H'' (and thus it also reflects the difference between the estimate of π_2^E under H' and the corresponding estimate of π_2^E under H''). This comment about the difference between the endowment distributions under model H' and under model H'' will be clarified further with formulae in $(A.11)$ later herein.

Returning now to our basic formula (3) for the latent class model for manifest variables A and B and latent variable X (and replacing the symbols S, M, and E used in the present section and in Figure 1 by the corresponding symbols A, B, and X used in formula (3)), we noted in footnote 2 that the latent class model described by formula (3) views the row variable A and the column variable B in a symmetric way (as was the case with Figure 1a, viewing row variable S and column variable M), while the equivalent latent class model, which was obtained simply by replacing $\pi_t^X \pi_{it}^{\bar{A}X}$ in formula (3) by the equivalent $\pi_i^A \pi_{ti}^{\bar{X}A}$, views these variables asymmetrically (as was the case with Figure 1b). (Recall that π_i^A here is the probability that an observation is in class i on variable A, and $\pi_{ti}^{\bar{X}A}$ is the conditional probability that an observation is in class t on variable X, given that the observation is in class i on variable A.) Similarly, another equivalent latent class model can be obtained simply by replacing $\pi_t^X \pi_{jt}^{\bar{B}X}$ in formula (3) by the equivalent $\pi_j^B \pi_{tj}^{\bar{X}B}$, where π_j^B is the probability that an observation is in class j on variable B, and $\pi_{tj}^{\bar{X}B}$ is the conditional probability that an observation is in class t on variable X, given that the observation is in class j on variable B.[19] The conditional probabilities $\pi_{ti}^{\bar{X}A}$ and $\pi_{tj}^{\bar{X}B}$ can be expressed simply as

$$\pi_{ti}^{\bar{X}A} = \pi_t^X \pi_{it}^{\bar{A}X} / \pi_i^A, \qquad \pi_{tj}^{\bar{X}B} = \pi_t^X \pi_{jt}^{\bar{B}X} / \pi_j^B. \tag{A.1}$$

(Note that the symbols π_i^A and π_j^B above have the same meaning as the symbols P_i^A and P_j^B in formulae (1) and (2) earlier herein.) The endowment distributions in Table A.3 present the estimates of $\pi_{ti}^{\bar{X}A}$ and $\pi_{tj}^{\bar{X}B}$ (i.e., the corresponding estimates of $\pi_{ti}^{\bar{E}S}$ and $\pi_{tj}^{\bar{E}M}$), under models H' and H'' (for $t = 1, 2$; $i = 1, \ldots, 6$; $j = 1, \ldots, 4$).

Having returned now to our basic formula (3) for the latent class model for manifest variables A and B and latent variable X (and replacing the symbols S, M, and E used earlier herein by the corresponding symbols A, B, and X used in formula (3)), we see that Tables A.1 and A.2 present the estimated $\pi_{it}^{\bar{A}X}$, $\pi_{jt}^{\bar{B}X}$, and π_t^X, under the equivalent latent class models H', H'', H''', and H''''; and Table A.3 presents the corresponding estimated $\pi_{ti}^{\bar{X}A}$ and $\pi_{tj}^{\bar{X}B}$, under the equivalent models H' and H''. We will now present some relatively simple *explicit* formulae that describe the relationship between the estimates of $\pi_{it}^{\bar{A}X}$ under two equivalent models (e.g., models H' and H''), between the estimates of $\pi_{jt}^{\bar{B}X}$ under the two models, between the estimates of π_t^X under the two models, between the estimates of $\pi_{ti}^{\bar{X}A}$ under the two models, and between the estimates of $\pi_{tj}^{\bar{X}B}$ under the two models.

With the latent class model described by formula (3), in the case where the latent variable is dichotomous we find that

$$\pi_{ij}^{AB} = \sum_{t=1}^{2} \pi_t^X \pi_{it}^{\bar{A}X} \pi_{jt}^{\bar{B}X}, \tag{$A.2$}$$

and

$$\pi_i^A = \sum_{t=1}^{2} \pi_t^X \pi_{it}^{\bar{A}X}, \qquad \pi_j^B = \sum_{t=1}^{2} \pi_t^X \pi_{jt}^{\bar{B}X}, \tag{$A.3$}$$

where π_{ij}^{AB} is the joint probability that an observation is in class i on variable A and in class j on variable B, where π_i^A is the probability that an observation is in class i on variable A, and where π_j^B is the probability that an observation is in class j on variable B. From $(A.2)$ and $(A.3)$, we find that

$$\pi_{ij}^{AB} - \pi_i^A \pi_j^B = \pi_1^X \pi_2^X \delta_i^A \delta_j^B, \tag{$A.4$}$$

where

$$\delta_i^A = \pi_{i1}^{\bar{A}X} - \pi_{i2}^{\bar{A}X}, \qquad \delta_j^B = \pi_{j1}^{\bar{B}X} - \pi_{j2}^{\bar{B}X}. \tag{$A.5$}$$

This set of formulae (viz. $(A.2)$–$(A.5)$) pertains to the estimates of the parameters (viz., π_t^X, $\pi_{it}^{\bar{A}X}$, $\pi_{jt}^{\bar{B}X}$) under the latent class model H (e.g., under model H'), and a corresponding set of formulae pertains to the estimates of the parameters (say, $\tilde{\pi}_t^X$, $\tilde{\pi}_{it}^{\bar{A}X}$, $\tilde{\pi}_{jt}^{\bar{B}X}$) under the equivalent latent class model $\tilde{H}$ (e.g., under model H''). Since the models H and $\tilde{H}$ are equivalent, we see from $(A.4)$ that

$$\gamma \delta_i^A \delta_j^B = \tilde{\gamma} \tilde{\delta}_i^A \tilde{\delta}_j^B, \tag{$A.6$}$$

where $\gamma = \pi_1^X \pi_2^X$ and $\tilde{\gamma} = \tilde{\pi}_1^X \tilde{\pi}_2^X$, and where $\tilde{\delta}_i^A$ and $\tilde{\delta}_j^B$ are defined using $\tilde{\pi}_{it}^{\bar{A}X}$ and $\tilde{\pi}_{jt}^{\bar{B}X}$ in the same way that δ_i^A and δ_j^B were defined in $(A.5)$ using $\pi_{it}^{\bar{A}X}$ and $\pi_{jt}^{\bar{B}X}$. We now define σ as the ratio $\tilde{\delta}_i^A / \delta_i^A$; and from $(A.6)$ we see that

$$\sigma = \tilde{\delta}_i^A / \delta_i^A = (\delta_j^B / \tilde{\delta}_j^B)(\gamma / \tilde{\gamma}). \tag{$A.7$}$$

Thus,

$$\tilde{\delta}_i^A = \delta_i^A \sigma, \quad \tilde{\delta}_j^B = \delta_j^B (\gamma/\tilde{\gamma})/\sigma. \tag{A.8}$$

From $(A.8)$ we see that the quantity σ can be viewed as a parameter that transforms the numerical value of δ_i^A (from δ_i^A to $\tilde{\delta}_i^A$) in direct proportion to the magnitude of σ, and that transforms the numerical value of δ_j^B (from δ_j^B to $\tilde{\delta}_j^B$) in inverse proportion to the magnitude of σ.[20] From $(A.8)$ we can see why I have called σ the multiplier parameter.

With the definition of σ introduced above, we can also obtain the following general formulae:[21]

$$\begin{aligned}(\tilde{\pi}_{it}^{\bar{A}X} - P_i^A) &= (\pi_{it}^{\bar{A}X} - P_i^A)[(\pi_t^X/\tilde{\pi}_t^X)/(\gamma/\tilde{\gamma})]\sigma, \\ (\tilde{\pi}_{jt}^{\bar{B}X} - P_j^B) &= (\pi_{jt}^{\bar{B}X} - P_j^B)(\pi_t^X/\tilde{\pi}_t^X)/\sigma.\end{aligned} \tag{A.9}$$

Note also that the formulae in $(A.9)$ can be rewritten and simplified as follows when $\tilde{\pi}_t^X = \pi_t^X$:[22]

$$\begin{aligned}\tilde{\pi}_{it}^{\bar{A}X} &= P_i^A + (\pi_{it}^{\bar{A}X} - P_i^A)\sigma, \\ \tilde{\pi}_{jt}^{\bar{B}X} &= P_j^B + (\pi_{jt}^{\bar{B}X} - P_j^B)/\sigma.\end{aligned} \tag{A.10}$$

When $\tilde{\pi}_t^X = \pi_t^X$, we see from $(A.10)$ that $(\pi_{it}^{\bar{A}X} - P_i^A)$ is transformed directly into $(\tilde{\pi}_{it}^{\bar{A}X} - P_i^A)$ using the multiplier parameter σ, and that $(\pi_{jt}^{\bar{B}X} - P_j^B)$ is transformed directly into $(\tilde{\pi}_{jt}^{\bar{B}X} - P_j^B)$ using $1/\sigma$ as the multiplier. We also see from $(A.10)$ that, when $\tilde{\pi}_t^X = \pi_t^X$, the $\tilde{\pi}_{it}^{\bar{A}X}$ can be expressed simply as a linear function of the multiplier parameter σ, and the $\tilde{\pi}_{jt}^{\bar{B}X}$ can be expressed similarly as a linear function of $1/\sigma$.[23]

Returning now for a moment to the more general formulae in $(A.9)$, we see that $(\pi_{it}^{\bar{A}X} - P_i^A)\pi_t^X/\gamma$ is transformed there directly into $(\tilde{\pi}_{it}^{\bar{A}X} - P_i^A)\tilde{\pi}_t^X/\tilde{\gamma}$ using the multiplier parameter σ, and that $(\pi_{jt}^{\bar{B}X} - P_j^B)\pi_t^X$ is transformed there directly into $(\tilde{\pi}_{jt}^{\bar{B}X} - P_j^B)\tilde{\pi}_t^X$ using $1/\sigma$ as the multiplier. Note that $(\pi_{it}^{\bar{A}X} - P_i^A)\pi_t^X/\gamma$ and $(\pi_{jt}^{\bar{B}X} - P_j^B)\pi_t^X$ can be expressed as $(\pi_{it}^{AX} - P_i^A\pi_t^X)/\gamma$ and $(\pi_{jt}^{BX} - P_j^B\pi_t^X)$, respectively, where π_{it}^{AX} denotes the joint probability that an observation will be in class i on variable A and in class t on variable X under the latent class model H, and where π_{jt}^{BX} denotes the corresponding joint probability pertaining to class j on variable B and class t on variable X under model H. We can thus rewrite the formulae in $(A.9)$ and $(A.10)$ to see how $(\pi_{it}^{AX} - P_i^A\pi_t^X)/\gamma$ and $(\pi_{jt}^{BX} - P_j^B\pi_t^X)$ are transformed into the corresponding $(\tilde{\pi}_{it}^{AX} - P_i^A\tilde{\pi}_t^X)/\tilde{\gamma}$ and

$(\tilde{\pi}_{jt}^{BX} - P_j^B \tilde{\pi}_t^X)$ under the equivalent latent class model $\tilde{H}$. We also note that $(\pi_{it}^{AX} - P_i^A \pi_t^X)/\gamma$ and $(\pi_{jt}^{BX} - P_j^B \pi_t^X)$ can be expressed as $(\pi_{ti}^{\bar{X}A} - \pi_t^X)P_i^A/\gamma$ and $(\pi_{tj}^{\bar{X}B} - \pi_t^X)P_j^B$, respectively, where $\pi_{ti}^{\bar{X}A}$ and $\pi_{tj}^{\bar{X}B}$ denote the conditional probabilities considered in $(A.1)$. And we can thus also rewrite the formulae in $(A.9)$ and $(A.10)$ to see how $(\pi_{ti}^{\bar{X}A} - \pi_t^X)/\gamma$ and $(\pi_{tj}^{\bar{X}B} - \pi_t^X)$ are transformed into the corresponding $(\tilde{\pi}_{ti}^{\bar{X}A} - \tilde{\pi}_t^X)/\tilde{\gamma}$ and $(\tilde{\pi}_{tj}^{\bar{X}B} - \tilde{\pi}_t^X)$ under the equivalent latent class model $\tilde{H}$. With this rewriting of the formulae in $(A.9)$ and $(A.10)$, we thus obtain the following formulae corresponding to $(A.9)$; viz.,

$$\begin{aligned} \tilde{\pi}_{ti}^{\bar{X}A} &= \tilde{\pi}_t^X + [\pi_{ti}^{\bar{X}A} - \pi_t^X](\tilde{\gamma}/\gamma)\sigma, \\ \tilde{\pi}_{tj}^{\bar{X}B} &= \tilde{\pi}_t^X + [\pi_{tj}^{\bar{X}B} - \pi_t^X]/\sigma; \end{aligned} \qquad (A.11)$$

and we obtain the following formulae corresponding to $(A.10)$ when $\tilde{\pi}_t^X = \pi_t^X$:

$$\begin{aligned} \tilde{\pi}_{ti}^{\bar{X}A} &= \pi_t^X + (\pi_{ti}^{\bar{X}A} - \pi_t^X)\sigma, \\ \tilde{\pi}_{tj}^{\bar{X}B} &= \pi_t^X + (\pi_{tj}^{\bar{X}B} - \pi_t^X)/\sigma. \end{aligned} \qquad (A.12)$$

From formulae in $(A.10)$ and $(A.12)$, we see what are the effects of the multiplier parameter σ when $\tilde{\pi}_t^X = \pi_t^X$; and from formulae in $(A.9)$ and $(A.11)$, we see what are the effects of σ and of $\tilde{\pi}_t^X$ when $\tilde{\pi}_t^X$ may differ from π_t^X.[24]

As we have seen here, in the analysis of the data in the two-way table (the 6×4 table—Table 1), we can estimate the parameters in the latent class model H_1 (a two-class latent class model—see Table 2) after two of the parameters in the model have been specified; and it is possible to specify the two parameters in the model in a way that is meaningful in the substantive context pertaining to the data. Next we return to the analysis of the data in the four-way table (the $2 \times 2 \times 2 \times 2$ table—Table 4). For these data, we shall now see that we can estimate the parameters in the latent class model M_2 (a three-class latent class model—see Tables 5.1 and 5.2) after one of the parameters in the model has been specified; and it is possible to specify this one parameter in the model in a way that is meaningful in the substantive context pertaining to these data.[25]

Model M_2 in Tables 5.1 and 5.2 is very different from the latent class model M_1 (a two-class latent class model) in Table 5.1 and in Table 6; and it is also very different from the other three-class latent class models in Table 5.2, viz., models M_3, M_4, and M_5. In model M_3, we see from Table 7 that the parameters that were specified were

$$\pi_{11}^{\bar{A}V} = \pi_{11}^{\bar{B}V} = \pi_{11}^{\bar{C}V} = \pi_{11}^{\bar{D}V} = 1 \text{ and } \pi_{13}^{\bar{A}V} = \pi_{13}^{\bar{B}V} = \pi_{13}^{\bar{C}V} = \pi_{13}^{\bar{D}V} = 0;$$

and this was also the case in models M_4 and M_5, with one more constraint (viz., $\pi_{12}^{\bar{B}V} = \pi_{12}^{\bar{C}V}$) introduced in model M_4, and two more constraints (viz., $\pi_{12}^{\bar{B}V} = \pi_{12}^{\bar{C}V}$ and $\pi_{12}^{\bar{A}V} = \pi_{22}^{\bar{D}V}$) introduced in model M_5.

Two different but equivalent versions of model M_2 are presented in Table A.4; viz., models M_2' and M_2''. The expected frequencies estimated under model M_2' for the cells in the four-way table will be equal to the corresponding expected frequencies estimated under model M_2''. In model M_2', there is a more stringent threshold specified for the particularistically inclined, with $\pi_{13}^{\bar{B}V}$ specified as zero; and in model M_2'', there is a more stringent threshold specified for the universalistically inclined, with $\pi_{11}^{\bar{C}V}$ specified as one. From Table A.4, we see that, under model M_2', the estimated values for $\pi_{13}^{\bar{A}V}$, $\pi_{13}^{\bar{B}V}$, $\pi_{13}^{\bar{C}V}$, and $\pi_{13}^{\bar{D}V}$ are .288, .000, .241, .057, respectively, with the more stringent threshold for the particularistically inclined; and, under model M_2'', the corresponding estimated values for $\pi_{21}^{\bar{A}V}$, $\pi_{21}^{\bar{B}V}$, $\pi_{21}^{\bar{C}V}$, and $\pi_{21}^{\bar{D}V}$ are .002, .020, .000, .087, respectively, with the more stringent threshold for the universalistically inclined. If we think of the responses to items A, B, C, and D as indicators or markers for the unobserved latent classes (i.e., the "true" latent types), where the unobserved variable is, in some sense, being measured (in an indirect way and with measurement error) by the observed variables, then the error rates for items A, B, C, and D are as noted above for the particularistically inclined under M_2'; and the corresponding error rates are also as noted above for the universalistically inclined under M_2''. The proportion of individuals in each of the three latent classes is estimated as $\pi_1^V = .22$, $\pi_2^V = .67$, and $\pi_3^V = .11$ under model M_2'; and the corresponding proportion is estimated as $\pi_1^V = .19$, $\pi_2^V = .58$, and $\pi_3^V = .23$ under model M_2''. As the proportion π_1^V decreases from .22 to .19 (i.e., from model M_2' to M_2''), the corresponding π_2^V also decreases, the corresponding π_3^V therefore increases, and all of the $\pi_{1t}^{\bar{A}V}$, $\pi_{1t}^{\bar{B}V}$, $\pi_{1t}^{\bar{C}V}$, and $\pi_{1t}^{\bar{D}V}$ increase. There is a one-dimensional continuum of models ranging between models M_2' and M_2'' (as

π_1^V ranges between .22 and .19) that will yield the same estimated expected frequencies as obtained under models M_2' and M_2''. Thus, even though the parameters in model M_2 can not be estimated until one of the parameters in the model is specified, by comparing the corresponding parameter values in models M_2' and M_2'' we can see what is the possible range of the parameter values; and we find that this range is relatively small here for most of the parameters in the model.

TABLE A.4 AROUND HERE

As we have noted above, when the three-class unrestricted latent class model M_2 is applied to the cross-classified data in Table 4 (the four-way $2 \times 2 \times 2 \times 2$ table), there is a one-dimensional continuum of models, ranging between models M_2' and M_2'', that will yield the same estimated expected frequencies as obtained under both models M_2' and M_2''; and when the two-class unrestricted latent class model H_1 is applied to the cross-classified data in Table 1 (the two-way 6×4 table), there is a two-dimensional continuum of models (contained within a subset of two-dimensional space), ranging between models H_1' and H_1'' and between models H_1''' and H_2'''', that will yield the same estimated expected frequencies as obtained under models H_1', H_1'', H_1''', and H_1''''. With respect to the possible range of the parameter values in the model under consideration, for some sets of data the range will be relatively small (as is the case for most of the parameters when model M_2 is applied to the data in Table 4) and for other sets of data this will not be the case.

As we have seen with the two examples considered in this Appendix, when the expected frequencies estimated under a particular model (say, model M) are congruent with the cross-classified data of interest, and this model is of the kind in which the parameters in the model can be estimated from the data only after some of the parameters are specified, it will often be possible to specify the appropriate parameters in a way that is meaningful in the substantive context pertaining to the data. Also, as we have seen with these two examples, it will often be possible to specify the appropriate parameters in different ways that yield the same estimated expected frequencies but provide opposite extremes with respect to some meaningful dimensions; thus obtaining, say, models M' and M'' when only one parameter needs to be specified, and models M', M'', M''', and M'''' when two parameters need to be specified, with M' and M'' providing opposite extremes with respect to one dimension, and M''' and M'''' providing opposite extremes with respect to a second dimension. Comparisons of the corresponding parameter values in models M' and M'' when only one parameter needs to be specified, or in models M', M'', M''', and M'''' when two parameters need to be specified, will shed additional light on the meaning of model M and on the cross-classified data of interest.

NOTES

1. The material presented in this exposition is a further development of some of the material presented by the author in the Clifford C. Clogg Memorial Lecture at the International Social Science Methodology Conference, held at the University of Essex in Colchester, England, in July 1996, and also a further development of some of the material presented by the author in the Keynote Address at the Conference on Social Science and Statistics, held in honor of the late Clifford C. Clogg at the Pennsylvania State University in September 1996.

2. Figure 1a can be used to describe the latent class model expressed by formula (3) in the preceding section (with A, B, and X in formula (3) replaced now by S, M, and E, respectively); and, similarly, Figure 1b can be used to describe the equivalent latent class model in which $\pi_t^X \pi_{it}^{\bar{A}X}$ in formula (3) is replaced simply by the equivalent $\pi_i^A \pi_{ti}^{\bar{X}A}$, where π_t^A here is the probability that an observation is in class i on variable A, and $\pi_{ti}^{\bar{X}A}$ is the conditional probability that an observation is in class t on variable X, given that the observation is in class i on variable A; and Figure 1c can be used to describe the equivalent latent class model in which $\pi_t^X \pi_{it}^{\bar{A}X}$ in formula (3) is replaced simply by the equivalent π_{it}^{AX}, where π_{it}^{AX} is the joint probability that an observation is in class i on variable A and in class t on variable X. (Note that the symbol π_i^A above has the same meaning as the symbol P_i^A in formulae (1) and (2) earlier herein.) Figure 1a views the row variable S and the column variable M in a symmetric way, while Figures 1b and 1c view these variables asymmetrically. (With Figure 1b, we presented earlier an asymmetric view of the latent class model that is sometimes nowadays referred to as the latent budget model; see, e.g., Goodman 1974a; Clogg 1981a; de Leeuw and van der Heijden 1988; van der Ark and van der Heijden 1998; and van der Ark 1999.)

3. The four questionnaire items pertain to the respondent's possible expectation (A) that a friend, who was in the respondent's car when the respondent (driving at least at 35 miles per hour in a 20 mile per hour zone) hit a pedestrian, would "testify under oath that the car speed was only 20 miles per hour" (which, according to the respondent's lawyer, "might save the respondent from serious consequences"); and (B) that a doctor friend, who works for an insurance company and examines the respondent when the respondent needs insurance and finds that he has some doubts on some minor points that are difficult to diagnose (in his examination of the respondent), would "shade the doubts in favor of the respondent"; and (C) that a drama-critic friend, who is reviewing a play that he (the drama critic) really thinks is no good, a play in which the respondent has sunk all his savings, would "go easy in his review of the respondent's play"; and (D) that a friend, who is a member of the board of directors of a company and has secret company information that could be financially ruinous to the respondent (unless the respondent gains this information), would "tip off the respondent".

4. Formula (12) can be used to estimate π_t^X by replacing P_{tt} and $\pi_{t,t,I+1}^{ABX}$ in this formula by the corresponding estimates described above (viz., p_{tt} and $\hat{\pi}_{t,t,I+1}^{ABX}$) when $p_{tt} \geq \hat{\pi}_{t,t,I+1}^{ABX}$, for $t = 1, 2, \ldots, I$. When this inequality is not satisfied for some values of t (say, for $t = t_0$), then the corresponding estimates of P_{tt} (for $t = t_0$) and $\pi_{t,t,I+1}^{ABX}$ (for $t = 1, 2, \ldots, I$) need to be modified in a straightforward way. The estimates of P_{tt} and $\pi_{t,t,I+1}^{ABX}$ in formula (12) can also be obtained directly from the observed cross-classified data in the $I \times I$ table using the $(I+1)$-class latent class model with appropriate restrictions (of the kind described above) imposed on the first I latent classes (see, e.g., restrictions (7.1)–(7.2) for $I = 2$).

5. For the m-way table (with $m \geq 2$), we could also consider more general latent class models in the situation in which there is a need for special treatment for, say, S specified cells in the table. Instead of the $(S+1)$-class latent class model with appropriate restrictions imposed on the first S latent classes (which correspond to the S specified cells in which there is a need for special treatment), we could consider, say, the more general $(S+2)$-class latent class model with appropriate restrictions imposed on the first S latent classes; and, instead of the corresponding m-way quasi-independence model, we would then have the more general m-way quasi-latent-structure model (with two latent classes rather than with one such latent class)

applied to the cross-classified data in the m-way table with the entries deleted in the S specified cells of the table (see, e.g., Goodman 1974a; Clogg 1981a).

6. For comments on the history and the prehistory (in the twentieth century and the nineteenth century) of work on the problem of measuring the relationship (the non-independence) between two or more observed dichotomous or polytomous variables, see, e.g., Goodman and Kruskal (1979), Stigler (1986), and Goodman (2000).

7. On the other hand, it might also be worthwhile to note here (as we noted in the introductory section of this exposition) that some mathematical models that were used earlier (in some nineteenth century and early twentieth century work) can now be viewed as special cases of latent class models or of other kinds of latent structures. The work by C. S. Peirce (1884) was considered, for example, in the preceding section; and, for other early references, see, e.g., Cournot (1838), Weinberg (1902), Benini (1928), and De Meo (1934).

8. The methods suggested by Green (1951) and by Gibson (1951) were described also in Anderson (1959); and the method suggested by Gibson (1951) was also described and applied in Lazarsfeld and Henry (1968).

9. Madansky (1968) showed that estimators that are not consistent can be obtained with a version of Green's method in a special case of the two-class latent class model applied to the three-way $2 \times 2 \times 2$ table; but the version of Green's method that was used and reported in Madansky (1968) considered the result that is obtained when the initial estimates suggested by Green are used without iteration (rather than the result that is obtained after the method is used in an iterative manner as described in Green 1951). However, Madansky has informed the author that he has now applied Green's method, with the iterative procedure described by Green, to the three-way table example used in Madansky (1968), and the iterative procedure does not converge in this case. A systematic investigation of the conditions under which convergence is obtained with the iterative procedure, and of the conditions under which consistent estimators are obtained, has not been carried out. Also, with respect to the application of Green's method to the m-way 2^m table, it should be noted that a particular difficulty inherent in this method (viz., the need to estimate the "elements with recurring subscripts" described in Green 1951) will be more

pronounced, in a certain sense, when $m = 3$, 4, and 5, and it will become less pronounced as m increases for $m \geq 5$. (The ratio of the number of elements with recurring subscripts to the number of elements with non-recurring subscripts decreases as m increases for $m \geq 3$; and the difference between the number of elements with non-recurring subscripts and the number of elements with recurring subscripts increases as m increases for $m \geq 4$, while this difference for $m = 3$ is equal to the difference for $m = 5$.) In commenting on Green's method, Lazarsfeld and Henry (1968) state that "...we do not think that it is a workable estimation method in practice."

10. For example, when Gibson's 1951 method is applied using a two-class latent class model in analyzing a four-way $2 \times 2 \times 2 \times 2$ table, the estimates that are actually obtained (for the parameters in the model) will depend on the particular angle of rotation that is selected with this method (see, e.g., the application of Gibson's method in Lazarsfeld and Henry, 1968, Sections 5.1–5.2). However, since the parameters in the two-class latent class model are identifiable (under certain conditions) when applied to the four-way $2 \times 2 \times 2 \times 2$ table (see, e.g., model M_1 in Table 5.1 and Table 6 earlier herein), the fact that Gibson's method would yield a range of estimates (depending on the angle of rotation that is used) in this case, indicates that Gibson's estimators are not consistent in this case. With respect to Gibson's 1951 method, Lazarsfeld and Henry (1968) comment as follows: "As far as we know, there do not exist any computer programs which will carry out the rotations of factors required by Gibson's method." Also, with respect to the rotations required by Gibson's method, Anderson (1959) comments as follows: "The rotation is laborious; this is an art that factor analysts have developed. The final result depends on the centering [i.e., on some 'centering principles' suggested by Gibson]; and it is impossible to give any mathematical results about this."

11. Since all of the parameters in the latent class model are probabilities and conditional probabilities (viz., the π_t^X, and the $\pi_{it}^{\bar{A}X}, \pi_{jt}^{\bar{B}X}, \ldots$), all of these quantities must lie in the closed interval from zero to one; but, it is often the case that some of the estimates that are actually obtained using the determinantal equations method will lie outside this interval (i.e., these estimates will be less than zero or more than one), and are thus not permissible.

With respect to the "efficiency" of an estimator, this statistical term is used here to mean "asymptotic efficiency". Whenever the determinantal equations method is applicable (i.e., under certain conditions when all of the observed variables in the m-way cross-classification table are dichotomous), this method will always yield estimators that are not efficient *except* in one special case; viz., the special case of the two-class latent class model applied to the three-way table in which, under certain conditions, the method could yield efficient estimators (see, e.g., Goodman 1974b). Even in this special case, when the method could yield efficient estimators, some of the estimates that are actually obtained using this method may not be permissible as estimates of the corresponding parameters in the latent class model (so the permissibility problem, which is present when the determinantal equations method is applied more generally, is also present even in this special case).

12. In commenting on the estimation method introduced in Goodman (1974a, 1974b) for obtaining the maximum-likelihood estimates of the parameters in the more general latent class models, McCutcheon (1987) writes as follows: "Goodman's procedure is simpler and more general [than earlier procedures]... Goodman's estimators provide a crucial breakthrough beyond the earlier approaches for the estimation of latent class parameters". McCutcheon (1987) also notes that "Goodman's estimation procedure has been implemented in a readily accessible computer program MLLSA (Maximum Likelihood Latent Structure Analysis)", and he refers the reader to Clogg (1977).

13. These simultaneous latent structure models can be viewed as a direct extension of the corresponding simultaneous loglinear models that were introduced earlier for the simultaneous analysis and comparison of the loglinear models pertaining to the cross-classified data in two or more multi-way tables; see, e.g., Goodman (1973a).

14. See, e.g., model 6 in Table 4 (with the corresponding formula (4.1), where the order of the variables in the model is C, A, B, D), and the related more general material on pp. 240–241, in Goodman (1970). In addition, Markov-type models were considered more explicitly and in more detail in Goodman (1971); and various generalizations of the Markov-type models were introduced in Goodman (1973b). For related material, see also, e.g., Anderson (1954b), Anderson and Goodman (1957), and Goodman (1962).

15. The iterative proportional fitting procedure in Haberman (1976), when it is used with the

general unconstrained latent class model, is equivalent to the estimation method introduced in Goodman (1974a,1974b); and the EM algorithm in Dempster, Laird, and Rubin (1977), when it is used with the unconstrained latent class model, is also equivalent to the estimation method introduced in Goodman (1974a,1974b). With respect to constrained latent class models and the particular kinds of equality constraints and fixed-value constraints that were described in Goodman (1974a, Sec. 5.2) for constraining the parameters in the model, the relatively simple procedures introduced in that article (for estimating the parameters in the corresponding constrained model) will provide maximum-likelihood estimates, as was noted in that article. (The particular kinds of constraints referred to above were also described in Goodman 1974b, Sec. 4.) With these particular kinds of constraints, we can obtain a wide range of useful models, as was demonstrated in Goodman (1974a,1974b). These kinds of constraints can also be generalized still further in a straightforward manner, and an even wider range of useful models is thereby obtained; but for this generalized set of constraints, the relatively simple iterative method introduced in Goodman (1974,1974b) would need to be modified somewhat. Mooijaart and van der Heijden (1992) present such a modified version; and a somewhat simpler modified version can also be obtained by using, as a part of the iterative method, an appropriate version of the uni-dimensional Newton algorithm introduced in Goodman (1979a); see Vermunt (1997,1999).

16. To simplify notation, we shall replace the notation for models H_1', H_1'', H_1''', and H_1'''' henceforth by H', H'', H''', and H'''', respectively.

17. With the asymmetric view presented in Figure 1b, the endowment distribution for those in each parental socioeconomic status category is directly relevant (and the distribution of mental health status for those in the favorably endowed latent class and for those in the not favorably endowed latent class is also directly relevant), while the corresponding endowment distribution for those in each mental health status category is not directly relevant. Nevertheless, for the sake of completeness, we have included in Table A.3 both the endowment distribution for those in each parental socioeconomic status category and the corresponding endowment distribution for those in each mental health status category.

18. In addition to the endowment distributions presented here under the latent class models

H' and H'', we could also have presented, for example, the corresponding endowment distributions under the latent class models H''' and H'''', which we considered earlier in Table A.2. This we leave for the interested reader.

19. The $\pi_t^X \pi_{jt}^{\bar{B}X}$ referred to above is obtained by rewriting the $\pi_t^X \pi_{it}^{\bar{A}X} \pi_{jt}^{\bar{B}X}$ in formula (3) simply as $\pi_t^X \pi_{jt}^{\bar{B}X} \pi_{it}^{\bar{A}X}$.

20. From the second formula in (A.8), we also see that $\tilde{\delta}_j^B / \delta_j^B$ is directly proportional to the magnitude of $\gamma/\tilde{\gamma}$ as well as being inversely proportional to the magnitude of σ.

21. The formulae in (A.9) can be obtained using the formulae in (A.3) pertaining to the estimates of the parameters (viz., π_t^X, $\pi_{it}^{\bar{A}X}$, $\pi_{jt}^{\bar{B}X}$) under the latent class model H, and the corresponding formulae pertaining to the estimates of the parameters (viz., $\tilde{\pi}_t^X$, $\tilde{\pi}_{it}^{\bar{A}X}$, $\tilde{\pi}_{jt}^{\bar{B}X}$) under the equivalent latent class model $\tilde{H}$. From the usual formulae for calculating the corresponding maximum-likelihood estimates (see, e.g., Goodman 1974a, 1974b), we find that the estimates of π_i^A and π_j^B in the formulae in (A.3) are equal to the corresponding estimates of P_i^A and P_j^B in the formulae in (1) and (2), viz., the corresponding observed proportion in class i on variable A and the corresponding observed proportion in class j on variable B.

22. The formulae in (A.10) and the more general formulae in (A.9) are, in some respects, simpler than some related (but different) formulae introduced in Goodman (1987); and the formulae presented here can help to shed additional light on the earlier results. For some other related (but quite different) kinds of results, see, e.g., Gilula (1983), van der Ark (1999), and van der Heijden, Gilula, and van der Ark (1999).

23. Since the probabilities $\tilde{\pi}_{it}^{\bar{A}X}$ and $\tilde{\pi}_{jt}^{\bar{B}X}$ in the formulae in (A.10), and in the more general formulae in (A.9), need to satisfy the usual inequality constraints for probabilities (viz., that $0 \leq \tilde{\pi}_{it}^{\bar{A}X} \leq 1$ and $0 \leq \tilde{\pi}_{jt}^{\bar{B}X} \leq 1$), the σ parameter is limited in its range (as we noted earlier herein). A similar result can also be obtained using the corresponding probabilities $\tilde{\pi}_{ti}^{\bar{X}A}$ and $\tilde{\pi}_{tj}^{\bar{X}B}$; see the corresponding formulae presented later herein in (A.12) and in (A.11). With respect to the linear function of the multiplier parameter σ and the linear function of $1/\sigma$ described by the formulae in (A.10), see also the corresponding linear functions in (A.12).

24. The formulae in (A.8) can be applied when $\tilde{\pi}_t^X = \pi_t^X$ and also when $\tilde{\pi}_t^X \neq \pi_t^X$.

25. With respect to the identifiability of the parameters in an unrestricted three-class latent class model applied to a $2 \times 2 \times 2 \times 2$ table, Lazarsfeld and Henry (1968, p. 65) present what they claim is a proof that the parameters in this case are unidentifiable; and Clogg (1981, p. 243), citing Lazarsfeld and Henry, repeats this claim. However, there is an error in the Lazarsfeld–Henry proof. (Their proof depends crucially on there being 15 basic parameters in the latent class model; but there actually are only $2 + (3 \times 4) = 14$ such parameters in this model.) For the cross-classified data in the particular $2 \times 2 \times 2 \times 2$ table, Table 4, Goodman (1974b) states that the parameters in the three-class latent class model are not "locally identifiable", and he explains why this is the case. (The reader should note that "identifiability" and "local identifiability" are different but related concepts. For further details, see, e.g., Goodman 1974b.)

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TABLE 1

Cross-Classification of 1,660 Subjects According to Their Parental Socioeconomic Status and Their Mental Health Status

Parental Socioeconomic Status		Mental Health Status: Well 1	Mild symptoms 2	Moderate symptoms 3	Impaired 4
High	1	64	94	58	46
	2	57	94	54	40
	3	57	105	65	60
	4	72	141	77	94
	5	36	97	54	78
Low	6	21	71	54	71

Source: Srole et al. (1962).

TABLE 2

Models Applied to the Cross-Classified Data in Table 1

Model	Number of Latent Classes	Degrees of Freedom	Goodness-of-Fit Chi-Square	Likelihood-Ratio Chi-Square
Independence model H_0	1	15	45.99	47.42
Latent class model H_1	2	8	2.74	2.75

TABLE 3

Observed Frequencies and Estimated Expected Frequencies under the Usual Null Model of Independence (H_0) and the Two-Class Latent Class Model (H_1), Applied to the Cross-Classified Data in Table 1; and the Corresponding Percentage Reduction in the Total Chi-Square

Cross-Classification	Observed Frequency	Estimated Expected Frequency: Model H_0	Model H_1
(1,1)	64	48.45	62.22
(1,2)	94	95.01	98.18
(1,3)	58	57.13	56.26
(1,4)	46	61.40	45.34
(2,1)	57	45.31	59.21
(2,2)	94	88.85	92.04
(2,3)	54	53.43	52.55
(2,4)	40	57.41	41.21
(3,1)	57	53.08	58.21
(3,2)	105	104.08	105.26
(3,3)	65	62.59	62.26
(3,4)	60	67.25	61.27
(4,1)	72	71.02	70.03
(4,2)	141	139.26	139.03
(4,3)	77	83.74	83.80
(4,4)	94	89.99	91.14
(5,1)	36	49.01	36.08
(5,2)	97	96.10	93.13
(5,3)	54	57.79	58.61
(5,4)	78	62.10	77.18
(6,1)	21	40.13	21.26
(6,2)	71	78.70	74.36
(6,3)	54	47.32	48.52
(6,4)	71	50.85	72.86
Goodness-of-fit chi-square		45.99	2.74
Percentage reduction		0%	94%
Degrees of freedom		15df	8df

TABLE 4

Cross-Classification of 216 Respondents According to Their Response in Four Different Situations of Role Conflict (Situations A, B, C, and D)

Response				Observed
A	B	C	D	Frequency
+	+	+	+	42
+	+	+	−	23
+	+	−	+	6
+	+	−	−	25
+	−	+	+	6
+	−	+	−	24
+	−	−	+	7
+	−	−	−	38
−	+	+	+	1
−	+	+	−	4
−	+	−	+	1
−	+	−	−	6
−	−	+	+	2
−	−	+	−	9
−	−	−	+	2
−	−	−	−	20

Source: Stouffer and Toby (1951, 1962, 1963).

Note: In each of the four different situations of role conflict, the response variable is dichotomous, and the respondent tends either toward *universalistic* values (+) or toward *particularistic* values (−) when responding to the situation with which he/she is confronted.

TABLE 5.1

Models Applied to the Cross-Classified Data in Table 4

Model	Number of Latent Classes	Degrees of Freedom	Goodness-of-Fit Chi-Square	Likelihood-Ratio Chi-Square
Independence model M_0	1	11	104.11	81.08
Latent class model M_1	2	6	2.72	2.72
Latent class model M_2	3	2	0.42	0.39

TABLE 5.2

More Parsimonious Models Applied to the Cross-Classified Data in Table 4

Model	Number of Latent Classes	Degrees of Freedom	Goodness-of-Fit Chi-Square	Likelihood-Ratio Chi-Square
Latent class model M_2	3	2	0.42	0.39
Latent class model M_3	3	9	2.28	2.28
Latent class model M_4	3	10	2.42	2.39
Latent class model M_5	3	11	2.85	2.72

TABLE 6

Distribution of Responses in Situation A, Responses in Situation B, Responses in Situation C, and Responses in Situation D, for Those in the Universalistically (+) Inclined Latent Class and for Those in the Particularistically (−) Inclined Latent Class, under the Two-Class Latent Class Model M_1 Applied to the Cross-Classified Data in Table 4

Response	Universalistically Inclined	Particularistically Inclined
Situation A		
+	.993	.714
−	.007	.286
Situation B		
+	.940	.330
−	.060	.670
Situation C		
+	.927	.354
−	.073	.646
Situation D		
+	.769	.132
−	.231	.868

Note: In model M_1, 28% are estimated to be in the universalistically inclined latent class, and 72% in the particularistically inclined class.

TABLE 7

Distribution of Responses in Situation A, Responses in Situation B, Responses in Situation C, and Responses in Situation D, for Those in the Strictly Universalistic (+) Latent Class, for Those in the Mixed Universalistically/Particularistically (+/−) Inclined Latent Class, and for Those in the Strictly Particularistic (−) Latent Class, under the Three-Class Latent Class Models M_3 Applied to the Cross-Classified Data in Table 4

Response	Strictly Universalistic	Universalistically/Particularistically Inclined	Strictly Particularistic
Situation A			
+	1.000	.796	.000
−	.000	.204	1.000
Situation B			
+	1.000	.420	.000
−	.000	.580	1.000
Situation C			
+	1.000	.437	.000
−	.000	.563	1.000
Situation D			
+	1.000	.175	.000
−	.000	.825	1.000

Note: In model M_3, 17% are estimated to be in the strictly universalistic latent class, 78% in the mixed universalistically/particularistically inclined class, and 5% in the strictly particularistic class.

TABLE 8

Distribution of Responses in Situation A, Responses in Situation B, Responses in Situation C, and Responses in Situation D, for Those in the Strictly Universalistic (+) Latent Class, for Those in the Mixed Universalistically/Particularistically (+/−) Inclined Latent Class, and for Those in the Strictly Particularistic (−) Latent Class, under the Three-Class Latent Class Models M_5 Applied to the Cross-Classified Data in Table 4

Response	Strictly Universalistic	Universalistically/Particularistically Inclined	Strictly Particularistic
Situation A			
+	1.00	.81	.00
−	.00	.19	1.00
Situation B			
+	1.00	.43	.00
−	.00	.57	1.00
Situation C			
+	1.00	.43	.00
−	.00	.57	1.00
Situation D			
+	1.00	.19	.00
−	.00	.81	1.00

Note: In model M_5, 17% are estimated to be in the strictly universalistic latent class, 77% in the mixed universalistically/particularistically inclined class, and 5% in the strictly particularistic class.

TABLE A.1

Distribution of Parental Socioeconomic Status and Distribution of Mental Health Status for Those in the Favorably Endowed Latent Class and for Those in the Not Favorably Endowed Latent Class, under Two Different but Equivalent Latent Class Models (Models H_1' and H_1'') Applied to the Cross-Classified Data in Table 1

Model H_1'	Favorably Endowed	Not Favorably Endowed
Parental socioeconomic status		
1	.25	.12
2	.24	.11
3	.21	.16
4	.22	.23
5	.07	.20
6	.00	.19
Mental health status		
1	.39	.10
2	.41	.34
3	.21	.22
4	.00	.34

Model H_1''	Favorably Endowed	Not Favorably Endowed
Parental socioeconomic status		
1	.20	.01
2	.19	.00
3	.19	.12
4	.23	.24
5	.12	.30
6	.07	.33
Mental health status		
1	.24	.00
2	.38	.32
3	.21	.23
4	.17	.45

Note: In the first model (H_1'), there is a stringent threshold for the favorably endowed; and in the second model (H_1''), there is a stringent threshold for the not favorably endowed. In model H_1', 30% are estimated to be in the favorably endowed latent class, and 70% in the not favorably endowed class; and in model H_1'', 23% are estimated to be in the not favorably endowed latent class, and 77% in the favorably endowed class.

TABLE A.2

Distribution of Parental Socioeconomic Status and Distribution of Mental Health Status for Those in the Favorably Endowed Latent Class and for Those in the Not Favorably Endowed Latent Class, under Two Additional Different but Equivalent Latent Class Models (Models H_1''' and H_1'''') Applied to the Cross-Classified Data in Table 1

Model H_1'''	Favorably Endowed	Not Favorably Endowed
Parental socioeconomic status		
1	.25	.01
2	.24	.00
3	.21	.12
4	.22	.24
5	.07	.30
6	.00	.33
Mental health status		
1	.24	.10
2	.38	.34
3	.21	.22
4	.17	.34

Model H_1''''	Favorably Endowed	Not Favorably Endowed
Parental socioeconomic status		
1	.20	.12
2	.19	.11
3	.19	.16
4	.23	.23
5	.12	.20
6	.07	.19
Mental health status		
1	.39	.00
2	.41	.32
3	.21	.23
4	.00	.45

Note: In model H_1''', the difference is maximized between the distribution of parental socioeconomic status for the favorably endowed and the corresponding distribution of parental socioeconomic status for the not favorably endowed; and in model H_1'''', the difference is maximized between the distribution of mental health status for the favorably endowed and the corresponding distribution of mental health status for the not favorably endowed. In model H_1''', 61% are estimated to be in the favorably endowed latent class, and 39% in the not favorably endowed class; and in model H_1'''', 48% are estimated to be in the favorably endowed latent class, and 52% in the not favorably endowed class.

TABLE A.3

Distribution of the Favorably Endowed and the Not Favorably Endowed for Those in Each Parental Socioeconomic Status Category and for Those in Each Mental Health Status Category, under Two Different but Equivalent Latent Class Models (Models H_1' and H_1'') Applied to the Cross-Classified Data in Table 1

Model H_1'	Favorably Endowed	Not Favorably Endowed
Parental socioeconomic status		
1	.48	.52
2	.50	.50
3	.36	.64
4	.29	.71
5	.13	.87
6	.00	1.00
Mental health status		
1	.63	.37
2	.34	.66
3	.28	.72
4	.00	1.00

Model H_1''	Favorably Endowed	Not Favorably Endowed
Parental socioeconomic status		
1	.98	.02
2	1.00	.00
3	.84	.16
4	.75	.25
5	.56	.44
6	.41	.59
Mental health status		
1	1.00	.00
2	.79	.21
3	.75	.25
4	.55	.45

Note: With respect to models H_1' and H_1'', see corresponding note below Table A.1.

TABLE A.4

Distribution of Responses in Situation A, Responses in Situation B, Responses in Situation C, and Responses in Situation D, for Those in the Universalistically (+) Inclined Latent Class, for Those in the Mixed Universalistically/Particularistically (+/−) Inclined Class, and for Those in the Particularistically (−) Inclined Latent Class, under Two Different but Equivalent Latent Class Models (the Three-Class Latent Class Models M_2' and M_2'') Applied to the Cross-Classified Data in Table 4

Model M_2' Response	Universalistically Inclined	Universalistically/Particularistically Inclined	Particularistically Inclined
Situation A			
+	.995	.806	.288
−	.005	.194	.712
Situation B			
+	.968	.428	.000
−	.032	.572	1.000
Situation C			
+	.976	.407	.241
−	.024	.593	.759
Situation D			
+	.863	.170	.057
−	.137	.830	.943

Model M_2'' Response	Universalistically Inclined	Universalistically/Particularistically Inclined	Particularistically Inclined
Situation A			
+	.998	.845	.483
−	.002	.155	.517
Situation B			
+	.980	.500	.096
−	.020	.500	.904
Situation C			
+	1.000	.448	.269
−	.000	.552	.731
Situation D			
+	.913	.202	.075
−	.087	.798	.925

Note: In model M_2', there is a more stringent threshold for the particularistically inclined; and in model M_2'', there is a more stringent threshold for the universalistically inclined. In model M_2', 22% are estimated to be in the universalistically inclined latent class, 67% in the mixed universalistically/particularistically inclined class, and 11% in the particularistically inclined class. In model M_2'', 19% are estimated to be in the universalistically inclined class, 58% in the mixed universalistically/particularistically inclined class, and 23% in the particularistically inclined class.

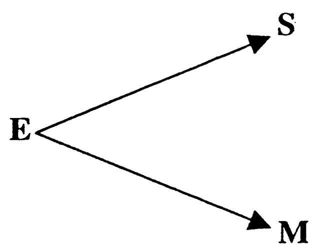

Fig. 1a: Explanatory latent variable E viewed as antecedent to variables S and M

Fig. 1b: Explanatory latent variable E' viewed as intervening between variables S and M

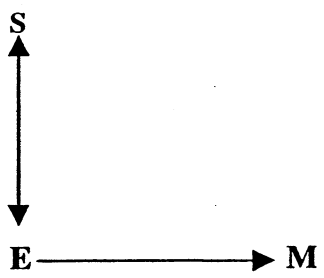

Fig. 1c: Explanatory latent variable E viewed as coincident or reciprocal with S and antecedent to M

Figure 1: Three different views of the relationship between variables S and M and variable E or variable E'

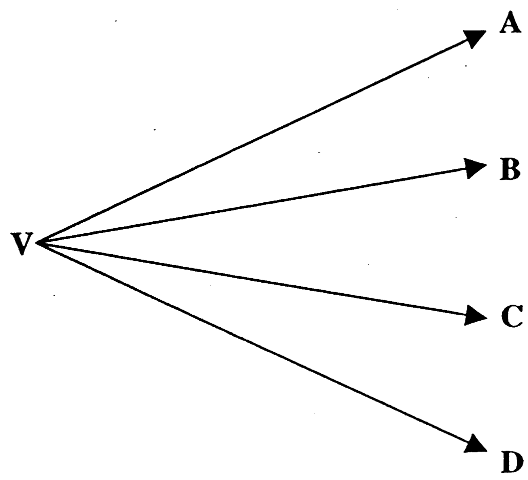

Figure 2: Explanatory latent variable *V* viewed as antecedent to response (manifest) variables *A, B, C,* and *D*